Interface problems for dam modeling

Thesis presented at the University of Montpellier
Doctoral school: Information Structures Systèmes

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Introduction

A posteriori error analysis

Constitutive relations for joints

Conclusions and perspectives

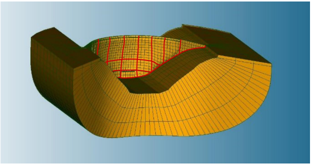
Motivation - Industrial context

- Finite element numerical simulations to study large hydraulic structures and evaluate their safety
- Complex behavior due to the combination of different effects (mechanical, thermal, hydraulic)
- Nonlinearity at the interface level
- Concrete dams show different interface zones:
  - concrete-rock contact in the foundation
  - joints between the blocks of the dam
  - joints in concrete
  - ...
- Gleno (Italy, 1923), Malpasset (France, 1959)

code_aster

Gleno

Malpasset



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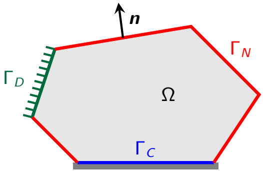
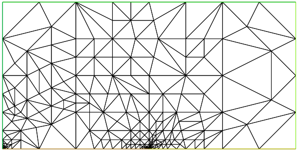
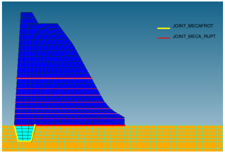
Contribution of the thesis

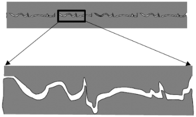
- Introduction of a posteriori error estimates for contact problems

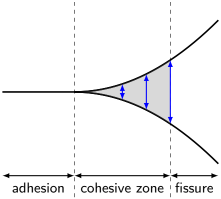




- Improvement of the current constitutive relations for joints (`JOINT_MECA_RUPT` and `JOINT_MECA_FROT`)







Introduction

A posteriori error analysis

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A posteriori estimate background

- System of PDEs with exact solution $\boldsymbol{u}$
- Numerical method $\Rightarrow$ approximate solution $\boldsymbol{u}_h$

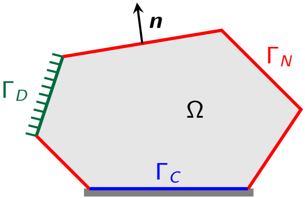
A posteriori error estimate:

$$|||\boldsymbol{u}-\boldsymbol{u}_h||| \leq \left(\sum_{T\in\mathcal{T}_h} \eta_T(\boldsymbol{u}_h)^2\right)^{1/2} \tag{1}$$

where $|||\cdot|||$ is some norm.

- Error control
- Local and global efficiency ($\eta_T(\boldsymbol{u}_h) \leq C\,|||\boldsymbol{u}-\boldsymbol{u}_h|||_{\mathcal{T}_T}$ for every element T)
- Error localization
- Identification and separation of different components of the error
- Adaptive mesh refinement (with some stopping criteria)

Unilateral contact problem

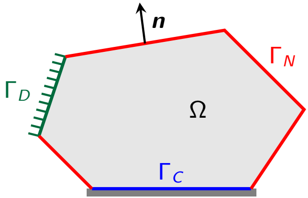


Strong formulation

$$\begin{aligned}
\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} &= \boldsymbol{0} && \text{in } \Omega, && \text{(2a)}\\
\boldsymbol{\sigma}(\boldsymbol{u}) &= \mathbb{E}\,\varepsilon(\boldsymbol{u}) && \text{in } \Omega, && \text{(2b)}\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_D, && \text{(2c)}\\
\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{g}_N && \text{on } \Gamma_N, && \text{(2d)}\\
u_n \leq 0,\ \sigma_n(\boldsymbol{u}) \leq 0,\ \sigma_n(\boldsymbol{u})\,u_n &= 0 && \text{on } \Gamma_C, && \text{(2e)}\\
\boldsymbol{\sigma_t}(\boldsymbol{u}) &= \boldsymbol{0} && \text{on } \Gamma_C. && \text{(2f)}
\end{aligned}$$

- $\boldsymbol{u}\colon \Omega(\subseteq \mathbb{R}^d) \to \mathbb{R}^d$, $d \in \{2,3\}$ is the unknown displacement
- $\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{ij}$, where $\varepsilon_{ij}(\boldsymbol{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, is the strain tensor
- $\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{E}\,\varepsilon(\boldsymbol{u}) := \lambda \mathrm{tr}\varepsilon(\boldsymbol{u})\boldsymbol{I}_d + 2\mu\varepsilon(\boldsymbol{u})$ is the elasticity stress tensor
- $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g}_N \in \boldsymbol{L}^2(\Gamma_N)$ are volume and surface forces, respectively
- $\boldsymbol{u} = u_n\boldsymbol{n} + \boldsymbol{u_t}$ and $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma_n(\boldsymbol{u})\boldsymbol{n} + \boldsymbol{\sigma_t}(\boldsymbol{u})$ on Γ_C

Unilateral contact problem



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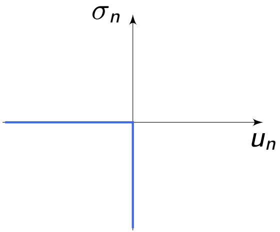

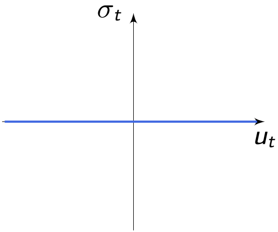

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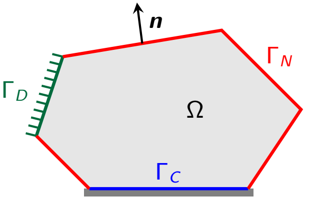

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\end{aligned}$$

$$\boldsymbol{H}^1_D(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \ :\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D \right\}$$

$$\boldsymbol{K} := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega) \ :\ v_n \leq 0 \text{ on } \Gamma_C \right\}$$

Weak formulation

Find $\boldsymbol{u} \in \boldsymbol{K}$ such that

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}), \varepsilon(\boldsymbol{v} - \boldsymbol{u})\big) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{g}_N, \boldsymbol{v} - \boldsymbol{u})_{\Gamma_N} \qquad \forall \boldsymbol{v} \in \boldsymbol{K}. \qquad (3)$$

Unilateral contact problem - Numerical approach

Let $\mathcal{T}_h$ be a triangulation of Ω, and $\boldsymbol{V}_h := \boldsymbol{H}^1_D(\Omega) \cap \boldsymbol{\mathcal{P}}^p(\mathcal{T}_h)$, $p \geq 1$. Moreover, we define $[\,\cdot\,]_{\mathbb{R}^-}$ as the projection on the half-line of negative real numbers $\mathbb{R}^-$, and the following operator

$$P_\gamma : \boldsymbol{V}_h \to L^2(\Gamma_C)$$
$$\boldsymbol{v}_h \mapsto \sigma_n(\boldsymbol{v}_h) - \gamma v_{h,n}.$$

The contact boundary condition (2e) can be rewritten as

$$\sigma_n(\boldsymbol{u}) = [P_\gamma(\boldsymbol{u})]_{\mathbb{R}^-} \,. \tag{4}$$

Nitsche-based method [Chouly-Hild2013]

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}_h), \varepsilon(\boldsymbol{v}_h)\big) - \Big(\big[P_\gamma(\boldsymbol{u}_h)\big]_{\mathbb{R}^-}, v_{h,n}\Big)_{\Gamma_C} = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{g}_N, \boldsymbol{v}_h)_{\Gamma_N} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Unilateral contact problem - Numerical approach

Nitsche-based method

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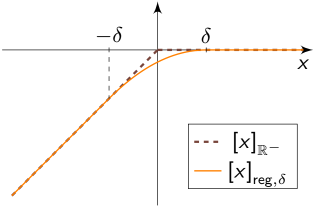
In order to solve this nonlinear problem

1. we regularize the projection operator $[\,\cdot\,]_{\mathbb{R}-}$ with $[\,\cdot\,]_{\text{reg},\delta}$,
2. we use Netwon method.



At each step $k \geq 1$ we have to solve the linear problem: Find $\boldsymbol{u}_h^k \in \boldsymbol{V}_h$ such that

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}_h^k), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) - \big(P_{\text{lin}}^{k-1}(\boldsymbol{u}_h^k), v_{h,n}\big)_{\Gamma_C} = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{g}_N, \boldsymbol{v}_h)_{\Gamma_N} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \quad (5)$$

A posteriori analysis - Measure of the error

At the k-th iteration of the Newton algorithm, we define the residual operator $\mathcal{R}(\boldsymbol{u}_h^k) \in (\boldsymbol{H}_D^1(\Omega))^*$ by

$$\langle \mathcal{R}(\boldsymbol{u}_h^k), \boldsymbol{v} \rangle := (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{g}_N, \boldsymbol{v})_{\Gamma_N} - \left(\boldsymbol{\sigma}(\boldsymbol{u}_h^k), \varepsilon(\boldsymbol{v})\right) + \left(\left[P_{1,\gamma}^n(\boldsymbol{u}_h^k)\right]_{\mathbb{R}^-}, v_n \right)_{\Gamma_C} \quad (6)$$

for all $\boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega)$. Then, the error between $\boldsymbol{u}$ and $\boldsymbol{u}_h^k$ is measured by the dual norm

$$\left\| \mathcal{R}(\boldsymbol{u}_h^k) \right\|_* := \sup_{\substack{\boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega), \\ \|\boldsymbol{v}\|_{C,h}=1}} \langle \mathcal{R}(\boldsymbol{u}_h^k), \boldsymbol{v} \rangle \quad (7)$$

where $\|\cdot\|_{C,h}$ is a norm which takes into account the contact boundary part:

$$\|\boldsymbol{v}\|_{C,h}^2 := \|\boldsymbol{\nabla} \boldsymbol{v}\|^2 + \sum_{F \in \mathcal{F}_h^C} \frac{1}{h_F} \|\boldsymbol{v}\|_F^2 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega). \quad (8)$$

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$\Rightarrow$ Comparison between the residual dual norm and the energy norm

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\text{en}}^2 = (\boldsymbol{\sigma}(\boldsymbol{u} - \boldsymbol{u}_h), \varepsilon(\boldsymbol{u} - \boldsymbol{u}_h)).$$

A posteriori analysis - Stress reconstruction

In general,

$$\boldsymbol{u}_h^k \in \boldsymbol{H}_D^1(\Omega) \qquad \text{but} \qquad \begin{cases} \boldsymbol{\sigma}(\boldsymbol{u}_h^k) \notin \mathbb{H}(\mathrm{div}, \Omega) \\ \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}_h^k) \neq -\boldsymbol{f} \\ \boldsymbol{\sigma}(\boldsymbol{u}_h^k)\boldsymbol{n} \neq \boldsymbol{g}_N \text{ on } \Gamma_N \end{cases}$$

where $\mathbb{H}(\mathrm{div}, \Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \mid \mathbf{div}\, \boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)\}$.

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where $\mathbb{H}(\mathrm{div}, \Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \mid \mathbf{div}\, \boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)\}$.

Stress reconstruction: $\begin{cases} \boldsymbol{\sigma}_h^k \in \mathbb{H}(\mathrm{div}, \Omega) \\ (\mathbf{div}\, \boldsymbol{\sigma}_h^k + \boldsymbol{f}, \boldsymbol{v}_T)_T = 0 & \forall \boldsymbol{v}_T \in \boldsymbol{\mathcal{P}}^0(T), \forall T \in \mathcal{T}_h \\ (\boldsymbol{\sigma}_h^k \boldsymbol{n}, \boldsymbol{v}_F)_F = (\boldsymbol{g}_N, \boldsymbol{v}_F)_F & \forall \boldsymbol{v}_F \in \boldsymbol{\mathcal{P}}^0(F), \forall F \in \mathcal{F}_h^N \end{cases}$

$$\boldsymbol{\sigma}_h^k = \boldsymbol{\sigma}_{h,\mathrm{dis}}^k + \underbrace{\boldsymbol{\sigma}_{h,\mathrm{reg}}^k}_{\text{regularization}} + \underbrace{\boldsymbol{\sigma}_{h,\mathrm{lin}}^k}_{\text{linearization}}$$

Local problems defined on patches using Arnold–Falk–Winther FE space. [Arnold-Falk-Winther2007]

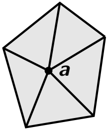

Figure: Patch around a vertex

$\Rightarrow$ Equilibrated, H-div conforming and weakly symmetric tensor $\boldsymbol{\sigma}_h^k$

Local estimators

- Stress estimator:

$$\boldsymbol{\sigma}_{h,\mathrm{dis}}^k \neq \boldsymbol{\sigma}(\boldsymbol{u}_h^k) \qquad \Rightarrow \qquad \eta_{\mathrm{str},T}^k := \|\boldsymbol{\sigma}_{h,\mathrm{dis}}^k - \boldsymbol{\sigma}(\boldsymbol{u}_h^k)\|_T$$

- Oscillation and Neumann estimators:

$$\mathbf{div}\,\boldsymbol{\sigma}_h^k \neq -\boldsymbol{f} \qquad \Rightarrow \qquad \eta_{\mathrm{osc},T}^k := \frac{h_T}{\pi}\left\|\boldsymbol{f} - \mathbf{div}\,\boldsymbol{\sigma}_h^k\right\|_T$$

$$\boldsymbol{\sigma}_h^k\boldsymbol{n} \neq \boldsymbol{g}_N \text{ on } \Gamma_N \qquad \Rightarrow \qquad \eta_{\mathrm{Neu},T}^k := \sum_{F\in\mathcal{F}_T^C} C_{t,T,F}\, h_F^{1/2}\left\|\boldsymbol{g}_N - \boldsymbol{\sigma}_h^k\boldsymbol{n}\right\|_F$$

- Contact estimator:

$$\sigma_{h,\mathrm{dis},n}^k \neq \left[P_\gamma(\boldsymbol{u}_h^k)\right]_{\mathbb{R}-} \qquad \Rightarrow \qquad \eta_{\mathrm{cnt},T}^k := \sum_{F\in\mathcal{F}_T^C} h_F^{1/2}\left\|\left[P_\gamma(\boldsymbol{u}_h^k)\right]_{\mathbb{R}-} - \sigma_{h,\mathrm{dis},n}^k\right\|_F$$

- Regularization and linearization estimators:

$$\eta_{\mathrm{reg1},T}^k := \|\boldsymbol{\sigma}_{h,\mathrm{reg}}^k\|_T \qquad \text{and} \qquad \eta_{\mathrm{reg2},T}^k := \sum_{F\in\mathcal{F}_T^C} h_F^{1/2}\left\|\sigma_{h,\mathrm{reg},n}^k\right\|_F$$

$$\eta_{\mathrm{lin1},T}^k := \|\boldsymbol{\sigma}_{h,\mathrm{lin}}^k\|_T \qquad \text{and} \qquad \eta_{\mathrm{lin2},T}^k := \sum_{F\in\mathcal{F}_T^C} h_F^{1/2}\left\|\sigma_{h,\mathrm{lin},n}^k\right\|_F$$

A posteriori analysis

Theorem (A posteriori error estimate)

$$\left\|\mathcal{R}(\boldsymbol{u}_h^k)\right\|_* \leq \left(\sum_{T\in\mathcal{T}_h}\left((\eta_{a,T}^k)^2+(\eta_{b,T}^k)^2\right)\right)^{1/2}$$

where

$$\eta_{a,T}^k := \eta_{\mathrm{osc},T}^k + \eta_{\mathrm{str},T}^k + \eta_{\mathrm{Neu},T}^k + \eta_{\mathrm{reg1},T}^k + \eta_{\mathrm{lin1},T}^k,$$

$$\eta_{a,T}^k := \eta_{\mathrm{cnt},T}^k + \eta_{\mathrm{reg2},T}^k + \eta_{\mathrm{lin2},T}^k.$$

Corollary (A posteriori error estimate)

$$\left\|\mathcal{R}(\boldsymbol{u}_h^k)\right\|_* \leq \left((\eta_a^k)^2+(\eta_b^k)^2\right)^{1/2}$$

where

$$\eta_a^k := \eta_{\mathrm{osc}}^k + \eta_{\mathrm{str}}^k + \eta_{\mathrm{Neu}}^k + \eta_{\mathrm{reg1}}^k + \eta_{\mathrm{lin1}}^k,$$

$$\eta_a^k := \eta_{\mathrm{cnt}}^k + \eta_{\mathrm{reg2}}^k + \eta_{\mathrm{lin2}}^k,$$

$$\eta_\bullet^k := \left(\sum_{T\in\mathcal{T}_h}\left(\eta_{\bullet,T}^k\right)^2\right)^{1/2}.$$

A posteriori analysis

Theorem (A posteriori error estimate)

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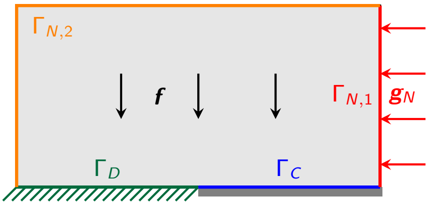
$$\eta_{a,T}^k := \eta_{\mathrm{cnt},T}^k + \eta_{\mathrm{reg2},T}^k + \eta_{\mathrm{lin2},T}^k.$$

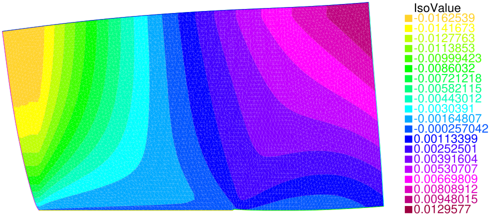
Adaptive algorithm

- Only the element where $\eta_{\mathrm{tot},T} := \left((\eta_{a,T}^k)^2 + (\eta_{b,T}^k)^2\right)^{1/2}$ is high are refined.
- The number of Newton iterations and the value of δ can be fixed automatically by the algorithm using some stopping criteria:

$$\eta_{\mathrm{reg1}}^k + \eta_{\mathrm{reg2}}^k \leq \gamma_{\mathrm{reg}}(\eta_{\mathrm{osc}}^k + \eta_{\mathrm{str}}^k + \eta_{\mathrm{Neu}}^k + \eta_{\mathrm{cnt}}^k + \eta_{\mathrm{lin1}}^k + \eta_{\mathrm{lin2}}^k), \tag{9}$$
$$\eta_{\mathrm{lin1}}^k + \eta_{\mathrm{lin2}}^k \leq \gamma_{\mathrm{lin}}(\eta_{\mathrm{osc}}^k + \eta_{\mathrm{str}}^k + \eta_{\mathrm{Neu}}^k + \eta_{\mathrm{cnt}}^k). \tag{10}$$

Numerical results





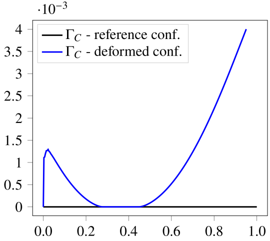
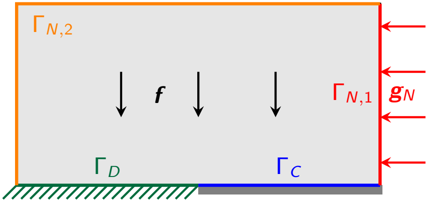
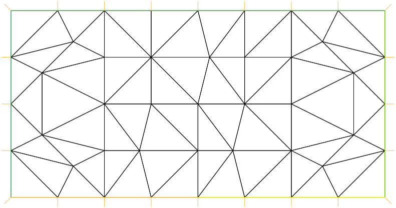
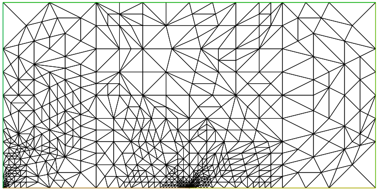


Figure: Vertical displacement in the deformed domain (amplification factor = 5): whole domain (*left*) and displacement of the contact boundary (*right*).

Adaptive mesh refinement

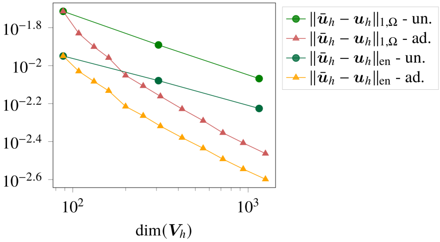


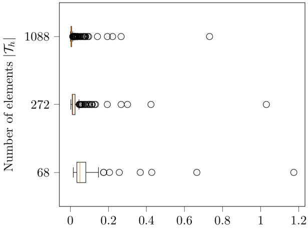


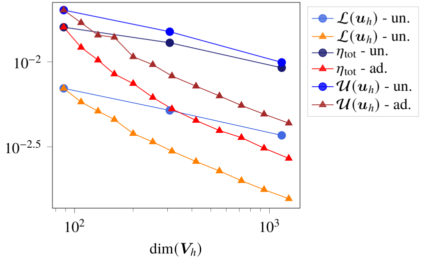


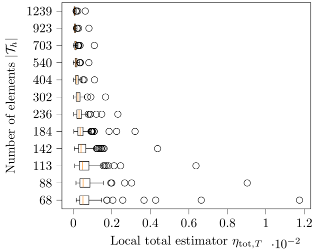
Adaptive VS Uniform refinement

$$\|\boldsymbol{v}\|_{\text{en}}^2 := (\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))$$









Stopping criteria

	Initial	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th
N_{reg}	7	0	1	0	0	0	0	0	0	0	0	0
N_{lin}	26	2	4	5	3	4	4	4	5	8	8	7

Table: Number of regularization iterations N_{reg} and Newton iterations N_{lin} at each refinement step of the adaptive algorithm with the stopping criteria (8) and (9).

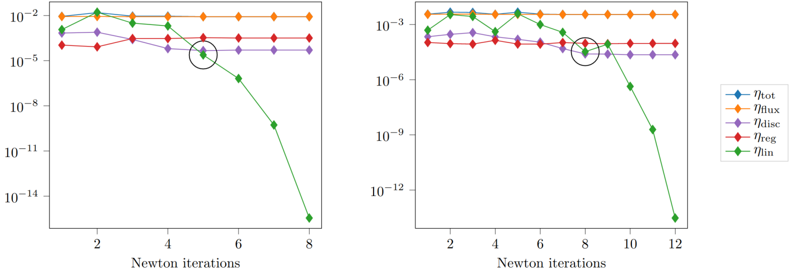


Figure: 3rd (*left*) and 9th (*right*) adaptively refined mesh

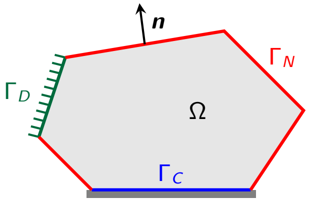
Introduction

A posteriori error analysis

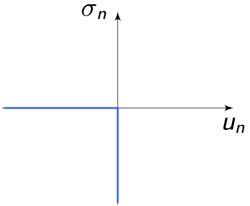
Constitutive relations for joints

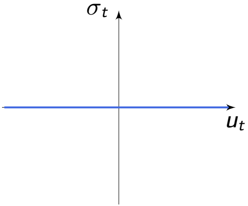
Conclusions and perspectives

Contact problem (without cohesive forces)



$$
\begin{aligned}
\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} &= \mathbf{0} && \text{in } \Omega, && \text{(11a)}\\
\boldsymbol{\sigma}(\boldsymbol{u}) &= \mathbb{E}\,\varepsilon(\boldsymbol{u}) && \text{in } \Omega, && \text{(11b)}\\
\boldsymbol{u} &= \mathbf{0} && \text{on } \Gamma_D, && \text{(11c)}\\
\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{g}_N && \text{on } \Gamma_N, && \text{(11d)}\\
\sigma_n(\boldsymbol{u}) &= [P_\gamma(\boldsymbol{u})]_{\mathbb{R}-} && \text{on } \Gamma_C, && \text{(11e)}\\
\boldsymbol{\sigma_t}(\boldsymbol{u}) &= \mathbf{0} && \text{on } \Gamma_C. && \text{(11f)}
\end{aligned}
$$

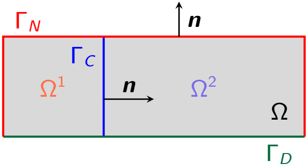




Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$
\big(\boldsymbol{\sigma}(\boldsymbol{u}_h), \varepsilon(\boldsymbol{v}_h)\big) - \left(\left[P_\gamma(\boldsymbol{u}_h)\right]_{\mathbb{R}-}, v_{h,n}\right)_{\Gamma_C} = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{g}_N, \boldsymbol{v}_h)_{\Gamma_N} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \quad (12)
$$

Joint problem with cohesive forces



$$\mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega \setminus \Gamma_C, \tag{13a}$$

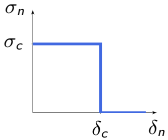
$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{E}\,\varepsilon(\boldsymbol{u}) \quad \text{in } \Omega \setminus \Gamma_C, \tag{13b}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{13c}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g}_N \quad \text{on } \Gamma_N, \tag{13d}$$

$$\boldsymbol{\sigma} = \boldsymbol{F}(\boldsymbol{\delta}) \quad \text{on } \Gamma_C, \tag{13e}$$

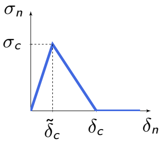
The displacement jump $\boldsymbol{\delta}$ and the force $(\boldsymbol{\sigma n} \equiv \boldsymbol{\sigma})$ between the two sides of the interface Γ_C are related through a **mechanical constitutive relation**.



Dugdale model

$$\boldsymbol{\delta} = -[\![\boldsymbol{u}]\!] := -(\boldsymbol{u}^1 - \boldsymbol{u}^2)$$

$$\boldsymbol{\delta} = (\delta_n, \delta_{t_1}, \delta_{t_2})^T$$



Bilinear model

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$(\boldsymbol{\sigma}(\boldsymbol{u}_h), \varepsilon(\boldsymbol{v}_h))_{\Omega\setminus\Gamma_C} + (\boldsymbol{F}(\boldsymbol{\delta}_h), \boldsymbol{\delta}_h^{\boldsymbol{v}})_{\Gamma_C} = (\boldsymbol{f}, \boldsymbol{v}_h)_{\Omega\setminus\Gamma_C} + (\boldsymbol{g}_N, \boldsymbol{v}_h)_{\Gamma_N} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{14}$$

where $\boldsymbol{\delta}_h := -[\![\boldsymbol{u}_h]\!]$ and $\boldsymbol{\delta}_h^{\boldsymbol{v}} := -[\![\boldsymbol{v}_h]\!]$.

- **Generalized standard materials** $\longrightarrow$ [Halphen-Quoc Son1975]

Geomaterials $\Rightarrow$ It establishes a class of elasto-plastic materials that satisfy the Clausius–Duhem inequality, and offers an energetic formulation for constructing a constitutive relation.

Adaptation to joint modeling $\Rightarrow$ **Variational framework** (Energy minimization) [Francfort-Marigo1998]

Ingredients (joint modeling):

- State variables $(\boldsymbol{\delta}, \boldsymbol{a})$
- Surface energy density $\psi(\boldsymbol{\delta}, \boldsymbol{a})$
- Reversibility domain $\mathbb{K}$ ($\Rightarrow$ Potential of dissipation $\phi(\boldsymbol{A})$)

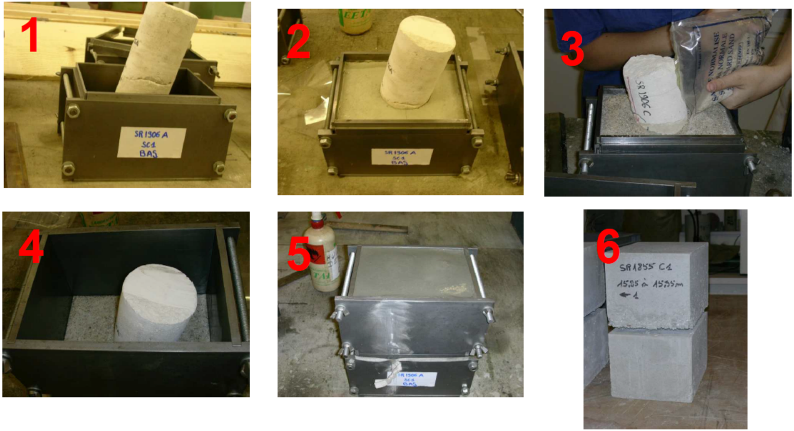
Features (joint modeling):

- The stress between the two sides of the interface and the thermodynamical internal forces are obtained by differentiation:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\delta}} \qquad \boldsymbol{A} = \frac{\partial \psi}{\partial \boldsymbol{a}}$$

- The reversibility domain $\mathbb{K}$ is convex
- The flow rule for $\boldsymbol{a}$ follows the normality rule

Example of shear test



Source: TEGG Lab - EDF [Unpublished]

Results of a shear test

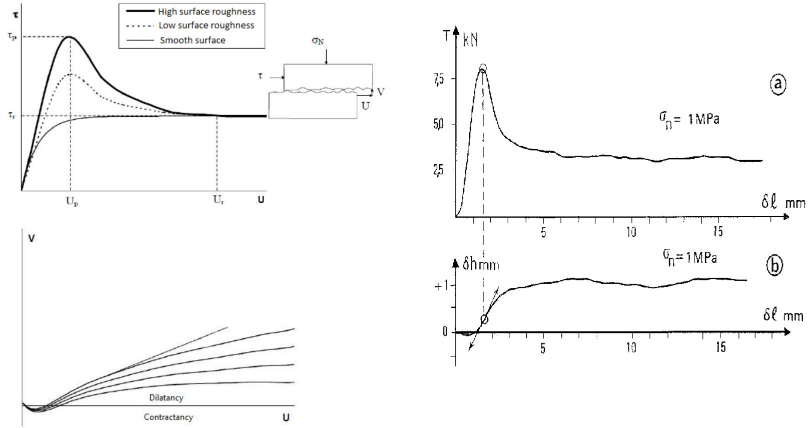
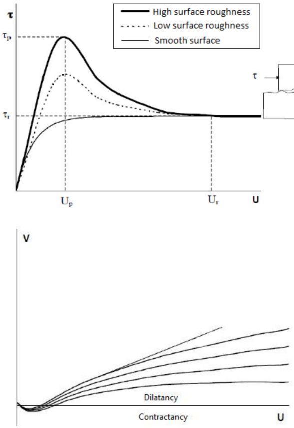


Figure: Typical curves of shear tests with fixed compression for joints: evolution of the shear stress (*top*) and of the normal displacement (*bottom*).

Results of a shear test



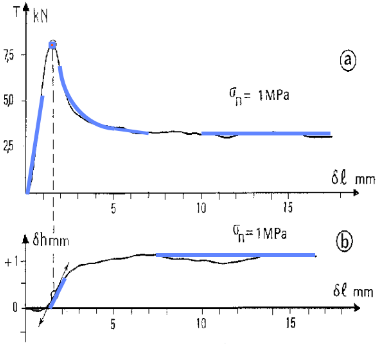
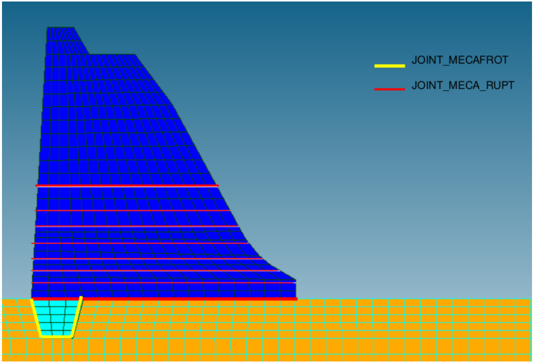


Figure: Typical curves of shear tests with fixed compression for joints: evolution of the shear stress (*top*) and of the normal displacement (*bottom*).

Existing constitutive relations in code_aster

- **JOINT_MECA_RUPT**: Rupture in traction
  - ▷ Rupture without plasticity!
- **JOINT_MECA_FROT**: Mohr–Coulomb non-associative standard law
  - ▷ Friction without rupture!



[R7.01.25] Lois de comportement des joints des barrages: JOINT_MECA_RUPT et JOINT_MECA_FROT, code_aster

Coupling plasticity and damage

Goals

- Phenomena to be reproduced: hardening/softening in traction and shear, dilatancy, ...
- To keep the normal flow rule for the evolution of plasticity
- To have a minimal number of parameters

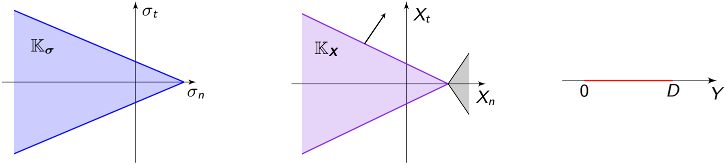
- State variables: displacement jump $\boldsymbol{\delta} \in \mathbb{R}^3$, plastic component $\boldsymbol{p} \in \mathbb{R}^3$, and damage variable $\alpha \in [0,1]$
- $\psi(\boldsymbol{\delta}, \boldsymbol{p}, \alpha)$ is the surface energy density function, convex with respect to $\boldsymbol{\delta}$, $\boldsymbol{p}$, α
- By differentiation, we obtain the thermodynamical forces related to the state variables:

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\delta}} \qquad \boldsymbol{X} = -\frac{\partial \psi}{\partial \boldsymbol{p}} \qquad Y = -\frac{\partial \psi}{\partial \alpha}$$

[Marigo-Kazymyrenko2019]

$$\psi(\boldsymbol{\delta}, \boldsymbol{p}, \alpha) = K_n \frac{(\delta_n - p_n)^2}{2} + K_t \frac{\|\boldsymbol{\delta_t} - \boldsymbol{p_t}\|^2}{2} + A_n(\alpha)\frac{(p_n)^2}{2} + A_t(\alpha)\frac{\|\boldsymbol{p_t}\|^2}{2}$$

- Kinematic hardening with the coupling of plasticity and damage, and damage functions defined by $A_s(\alpha) := B_s \frac{(1-\alpha)^{m_1}}{\alpha^{m_2}}$, $s \in \{n, t\}$
- Reversibility domains (fixed in the thermodynamical space):



$\boldsymbol{\sigma} = \boldsymbol{X} + \mathbb{A}(\alpha)\boldsymbol{p}$ $\quad \|\boldsymbol{X_t}\| + \mu X_n - c \leq 0$ $\quad 0 \leq Y \leq D$

- Irreversibility of damage ($\dot{\alpha} \geq 0$)
- Simultaneous evolution of $\boldsymbol{p}$ and α

$$\psi(\boldsymbol{\delta}, \boldsymbol{p}, \alpha) = \boxed{K_n \frac{(\delta_n - p_n)^2}{2} + K_t \frac{\|\boldsymbol{\delta_t} - \boldsymbol{p_t}\|^2}{2}} + A_n(\alpha)\frac{(p_n)^2}{2} + A_t(\alpha)\frac{\|\boldsymbol{p_t}\|^2}{2}$$

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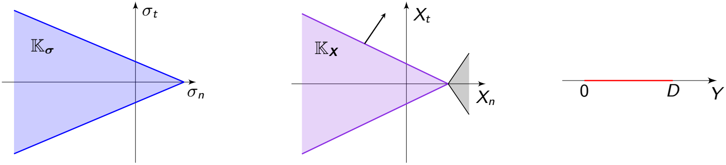

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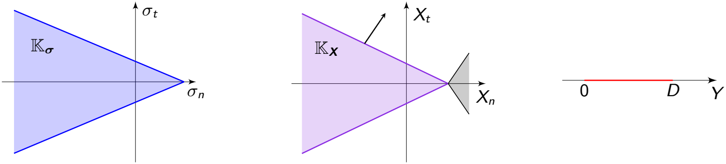

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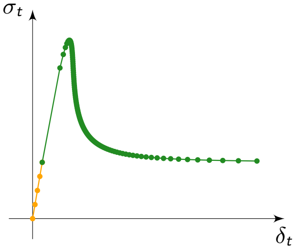
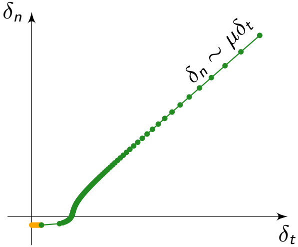
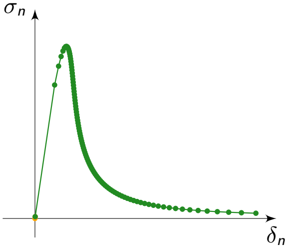
Numerical results

- Shear test with fixed compression



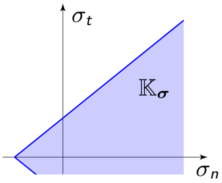


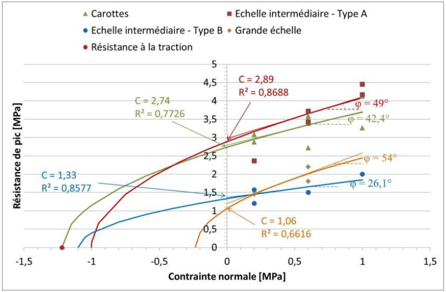
- Traction test



Some possible modifications

- Direct modification of the plasticity criterion ($\Rightarrow$ stress elastic domain $\mathbb{K}_{\boldsymbol{\sigma}}$)





[Mouzannar2016]

- Addition of hyperelasticity

$$K_n(\delta_n), K_n(\boldsymbol{\delta_t}), K_n(\delta_n, \boldsymbol{\delta_t}) \qquad K_t(\delta_n), K_t(\boldsymbol{\delta_t}), K_t(\delta_n, \boldsymbol{\delta_t})$$

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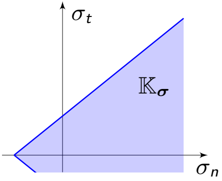

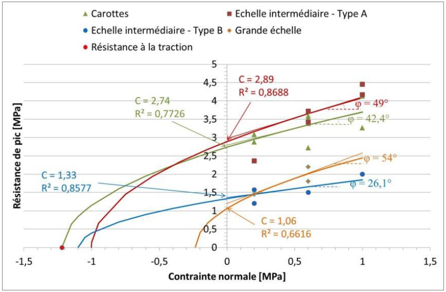

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$$\boxed{K_n(\delta_n)}, K_n(\boldsymbol{\delta_t}), K_n(\delta_n, \boldsymbol{\delta_t}) \qquad \boxed{K_t(\delta_n)}, K_t(\boldsymbol{\delta_t}), K_t(\delta_n, \boldsymbol{\delta_t})$$

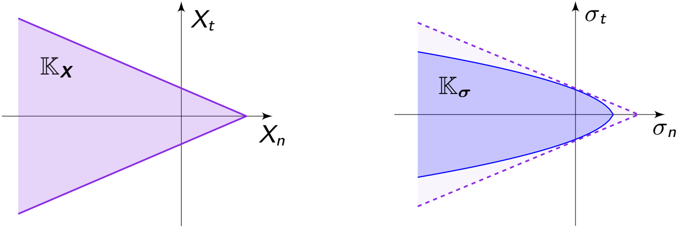
Model with hyperelasticity

$$\psi = K_n(\delta_n)\frac{(\delta_n - p_n)^2}{2} + K_t(\delta_n)\frac{\|\boldsymbol{\delta_t} - \boldsymbol{p_t}\|^2}{2} + A_n(\alpha)\frac{(p_n)^2}{2} + A_t(\alpha)\frac{\|\boldsymbol{p_t}\|^2}{2}$$

- Two new parameters: $\beta_n \geq 0$ and $\beta_t \geq 0$

$$K_s(\delta_n) := \frac{K_{s,0}}{2K_{s,0}\beta_s\delta_n + 1} \qquad s \in \{n, t\}$$

- Domain transformation:

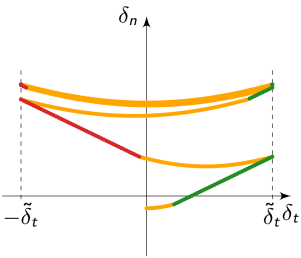


$$\|\boldsymbol{X_t}\| + aX_n - b \leq 0 \qquad \|\boldsymbol{\sigma_t}\| - \sqrt{A - B\sigma_n} - C \leq 0$$

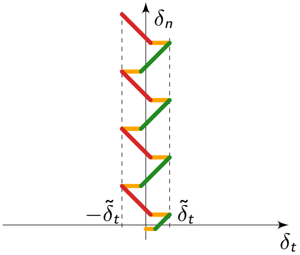
Model with hyperelasticity

$$\psi = K_n(\delta_n)\frac{(\delta_n - p_n)^2}{2} + K_t(\delta_n)\frac{\|\boldsymbol{\delta_t} - \boldsymbol{p_t}\|^2}{2} + A_n(\alpha)\frac{(p_n)^2}{2} + A_t(\alpha)\frac{\|\boldsymbol{p_t}\|^2}{2}$$

- Cyclic shear loading (asymptotic behavior, i.e., without damage):



With hyperelasticity

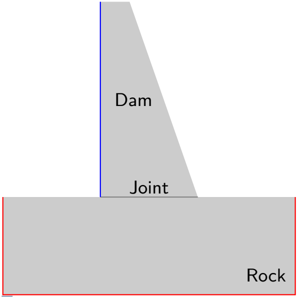


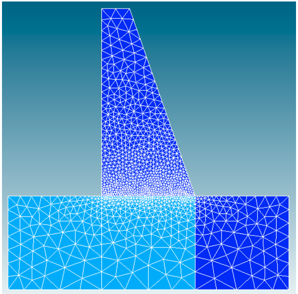
Without hyperelasticity

An example on a dam

$$\psi = \frac{K_{n,0}}{2K_{n,0}\beta_n\delta_n + 1}\frac{(\delta_n - p_n)^2}{2} + \frac{K_{t,0}}{2K_{t,0}\beta_t\delta_n + 1}\frac{\|\boldsymbol{\delta_t} - \boldsymbol{p_t}\|^2}{2}$$

We consider the 2D dam model shown by the figures (validation test `ssnp142a`): the height of the dam is 10 m, the length of the joint is 5 m, the length of the top part of the dam is 1.5 m, and the rock foundation has length 15 m and height 5 m.





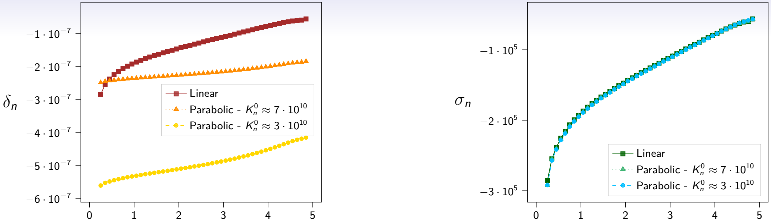


Figure: Vertical displacement δ_n (*left*) and normal stress σ_n (*right*) without lateral water pressure.

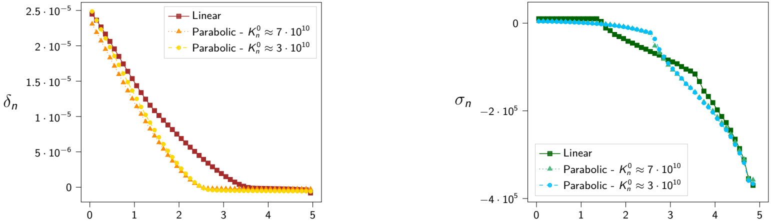


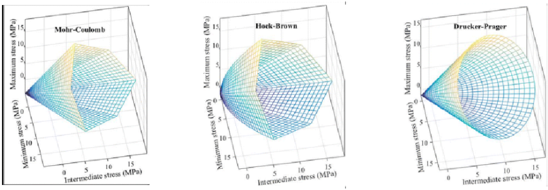
Figure: Vertical displacement δ_n (*left*) and normal stress σ_n (*right*) with lateral water pressure (9 meters) and imposed pressure inside of the joint.

Extension to geomaterials

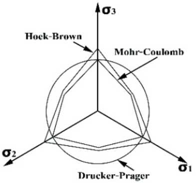
$$\phi(\varepsilon, \boldsymbol{p}) = \frac{1}{2}K(\mathrm{Tr}\varepsilon)(\mathrm{Tr}\varepsilon - \mathrm{Tr}\boldsymbol{p})^2 + \mu(\mathrm{Tr}\varepsilon)\|\varepsilon^D - \boldsymbol{p}^D\|^2$$

$$\boldsymbol{\sigma} = \boldsymbol{X} - \left(\beta_m(\underbrace{X_m}_{:=\mathrm{Tr}\boldsymbol{X}/3})^2 + \beta^D\|\underbrace{\boldsymbol{X}^D}_{:=\boldsymbol{X}-X_m\boldsymbol{I}_2}\|^2\right)$$

$$\frac{1}{\sqrt{6}}\|\boldsymbol{X}^D\| + aX_m - b \leq 0 \quad \Rightarrow \quad \frac{1}{\sqrt{6}}\|\boldsymbol{\sigma}^D\| - \sqrt{A - B\sigma_m} - C \leq 0$$



[Mehranpour *et al.*2016]



[Ghasempour *et al.*2017]

Conclusions:

- A posteriori estimate of the error measured with a dual norm for the contact problem without friction via stress reconstruction.
- We distinguish the different error components and we propose an adaptive algorithm with stopping criteria.
- Better asymptotic convergence with adaptive refinement.
- Joint model coupling plasticity and damage.
- Joint model with hyperelasticity: modification of the shape of plasticity criterion; stabilization of dilatancy in cycling loadings.

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Thank you for your attention!

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Equilibrated stress reconstruction

Find $(\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{r}_h^{\boldsymbol{a}}, \boldsymbol{\lambda}_h^{\boldsymbol{a}}) \in \boldsymbol{\Sigma}_{h,N,C}^{\boldsymbol{a}} \times \boldsymbol{U}_h^{\boldsymbol{a}} \times \boldsymbol{\Lambda}_h^{\boldsymbol{a}}$ such that:

$$(\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{\tau}_h)_{\omega_{\boldsymbol{a}}} + (\boldsymbol{r}_h^{\boldsymbol{a}}, \boldsymbol{div}\,\boldsymbol{\tau}_h)_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\lambda}_h^{\boldsymbol{a}}, \boldsymbol{\tau}_h)_{\omega_{\boldsymbol{a}}} = (\psi_{\boldsymbol{a}} \boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{\tau}_h)_{\omega_{\boldsymbol{a}}} \tag{15a}$$

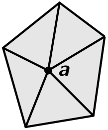
$$(\boldsymbol{div}\,\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} = (-\psi_{\boldsymbol{a}} \boldsymbol{f} + \boldsymbol{\sigma}(\boldsymbol{u}_h)\boldsymbol{\nabla}\psi_{\boldsymbol{a}}, \boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} \tag{15b}$$

$$(\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{\mu}_h)_{\omega_{\boldsymbol{a}}} = 0 \tag{15c}$$

for all $(\boldsymbol{\tau}_h, \boldsymbol{v}_h, \boldsymbol{\mu}_h) \in \boldsymbol{\Sigma}_h^{\boldsymbol{a}} \times \boldsymbol{U}_h^{\boldsymbol{a}} \times \boldsymbol{\Lambda}_h^{\boldsymbol{a}}$.

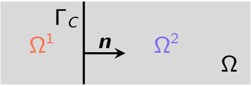
$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$

- $\boldsymbol{\Sigma}_h^{\boldsymbol{a}} := \{\boldsymbol{\tau}_h \in \mathbb{P}^p(\omega_{\boldsymbol{a}}) \cap \mathbb{H}(\mathbf{div}, \omega_{\boldsymbol{a}}) \ : \ \text{Hom. cond.}\}$
- $\boldsymbol{\Sigma}_{h,N,C}^{\boldsymbol{a}} := \{\boldsymbol{\tau}_h \in \mathbb{P}^p(\omega_{\boldsymbol{a}}) \cap \mathbb{H}(\mathbf{div}, \omega_{\boldsymbol{a}}) \ : \ \text{Non-hom./Hom. cond.}\}$
- $\boldsymbol{U}_h^{\boldsymbol{a}} := \boldsymbol{\mathcal{P}}^{p-1}(\omega_{\boldsymbol{a}}) \ / \ \boldsymbol{\mathcal{P}}^{p-1}(\omega_{\boldsymbol{a}}) \cap (\boldsymbol{RM}^d)^{\perp}$
- $\boldsymbol{\Lambda}_h^{\boldsymbol{a}} := \{\boldsymbol{\mu}_h \in \mathbb{P}^{p-1}(\omega_{\boldsymbol{a}}) \ : \ \boldsymbol{\mu}_h = -\boldsymbol{\mu}_h\}$



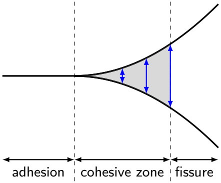
Constitutive relation for joints - Hypotheses

- **Cohesive zone model (CZM)** $\longrightarrow$ [Dugdale1960], [Barenblatt1962]



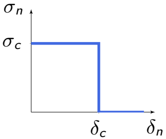
$$\boldsymbol{\delta} = -[\![\boldsymbol{u}]\!] := -(\boldsymbol{u}^1 - \boldsymbol{u}^2)$$

$$\boldsymbol{\delta} = (\delta_n, \delta_{t_1}, \delta_{t_2})^T$$

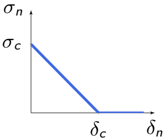


The displacement jump $\boldsymbol{\delta}$ and the force between the two sides of the interface Γ_C are related through a **mechanical constitutive relation**:

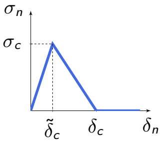
$$\boldsymbol{\sigma} = (\sigma_n, \sigma_{t_1}, \sigma_{t_2})^T = \boldsymbol{F}(\boldsymbol{\delta})$$



Dugdale model



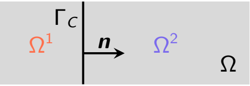
Softening linear model



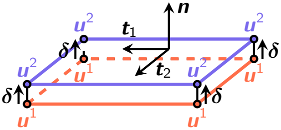
Bilinear model

Constitutive relation for joints - Hypotheses

- **Cohesive zone model (CZM)** $\longrightarrow$ [Dugdale1960], [Barenblatt1962]



$$\boldsymbol{\delta} = -[\![\boldsymbol{u}]\!] := -(\boldsymbol{u}^1 - \boldsymbol{u}^2)$$
$$\boldsymbol{\delta} = (\delta_n, \delta_{t_1}, \delta_{t_2})^T$$



The displacement jump $\boldsymbol{\delta}$ and the force between the two sides of the interface Γ_C are related through a **mechanical constitutive relation**:

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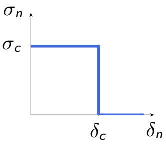
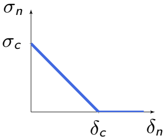
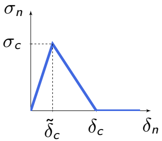


Dugdale model



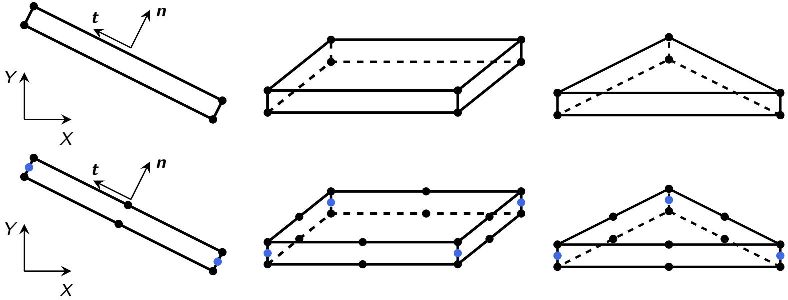
Softening linear model



Bilinear model

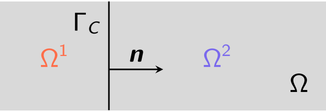
[R3.06.09] Élements finis de joint mécaniques et éléments finis de joint couplé hydromécanique, code_aster

Joint finite elements



Constitutive relation for joints - Hypotheses

- **Variational framework** $\longrightarrow$ Energy minimization [Francfort-Marigo1998]



$$\boldsymbol{\delta} = -[\![\boldsymbol{u}]\!] := -(\boldsymbol{u}^1 - \boldsymbol{u}^2)$$
$$\boldsymbol{\delta} = (\delta_n, \delta_{t_1}, \delta_{t_2})^T$$

$$\min_{\boldsymbol{u}} E_{\text{tot}}(\boldsymbol{u})$$
$$E_{\text{tot}}(\boldsymbol{u}) = E_{\text{tot}}(\boldsymbol{u}, \boldsymbol{\delta}) := E_{\text{el}}(\boldsymbol{u}) + E_{\text{sur}}(\boldsymbol{\delta}) - W_{\text{ext}}(\boldsymbol{u})$$

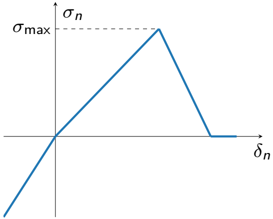
- $E_{\text{el}}(\boldsymbol{u}) := \int_{\Omega\setminus\Gamma} \phi(\varepsilon(\boldsymbol{u}))\,\mathrm{d}\Omega = \int_{\Omega\setminus\Gamma} \left(\frac{1}{2}\varepsilon(\boldsymbol{u})\,\mathbb{E}\,\varepsilon(\boldsymbol{u})\right)\mathrm{d}\Omega$ is the elastic energy,
- $E_{\text{sur}}(\boldsymbol{\delta}) := \int_{\Gamma} \psi(\boldsymbol{\delta})\,\mathrm{d}\Gamma$ is the surface energy,
- $W_{\text{ext}}(\boldsymbol{u}) = \int_{\Omega\setminus\Gamma} \boldsymbol{f}\,\boldsymbol{u}\,\mathrm{d}\Omega + \int_{\Gamma_N} \boldsymbol{g}_N\,\boldsymbol{u}\,\mathrm{d}\Gamma$ is the work of the external forces

$$F_{\text{int}}(\boldsymbol{u}, \boldsymbol{v}) = F_{\text{int}}(\boldsymbol{v}) \qquad \forall ť\boldsymbol{v}$$

Existing laws in code_aster

- **JOINT_MECA_RUPT**: Rupture in traction

$$\psi_n(\delta_n) = A(\delta_n)\psi_n^{\mathrm{con}}(\delta_n) + B(\delta_n)\psi_n^{\mathrm{lin}}(\delta_n) + C(\delta_n)\psi_n^{\mathrm{dis}}(\delta_n)$$



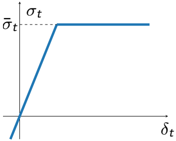
▷ Rupture without plasticity!

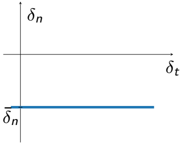
- **JOINT_MECA_FROT**: Mohr–Coulomb non-associative standard law

$$\psi(\boldsymbol{\delta}, \boldsymbol{p_t}) = \psi_n(\delta_n) + K_t \frac{\|\boldsymbol{\delta_t} - \boldsymbol{p_t}\|^2}{2}$$

Conic surface of charge du type Drucker–Prager (with possibly isotropic hardening):

$$\|\boldsymbol{\sigma_t}\| + \mu\sigma_n - c - K\lambda = 0$$





▷ Friction without rupture!

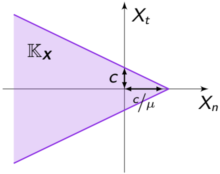
[R7.01.25] Lois de comportement des joints des barrages: JOINT_MECA_RUPT et JOINT_MECA_FROT, `code_aster`

Parameters fitting

$$\psi(\boldsymbol{\delta}, \boldsymbol{p}, \alpha) = K_n \frac{(\delta_n - p_n)^2}{2} + K_t \frac{\|\boldsymbol{\delta_t} - \boldsymbol{p_t}\|^2}{2} + A_n(\alpha)\frac{(p_n)^2}{2} + A_t(\alpha)\frac{\|\boldsymbol{p_t}\|^2}{2}$$

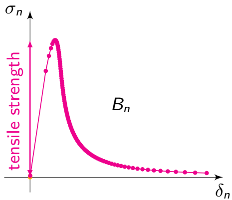
Parameters of the model: $K_n, K_t,$ | $B_n, B_t, m_1, m_2,$ | $\mu, c,$ | D

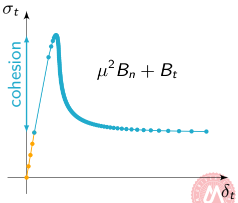
- $K_n > 0$ and $K_t > 0 \to$ normal and tangential rigidity
- $\mu > 0$ and $c \geq 0 \to$ shape of $\mathbb{K}_{\boldsymbol{X}}$ (friction coefficient and residual adhesion)



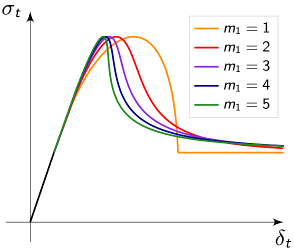
$$A_s(\alpha) = B_s \frac{(1-\alpha)^{m_1}}{\alpha^{m_2}}$$

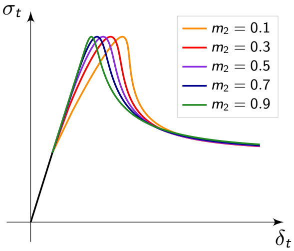
- B_n and $B_t \to$ peaks (tensile strength and cohesion)



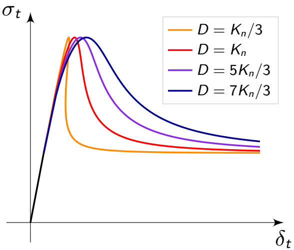


- $m_1 > 1$ and $0 < m_2 < 1 \rightarrow$ damage evolution





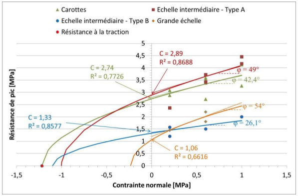
- $D \rightarrow$ snapback



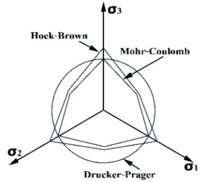
Evolution of the tangential stress in a shear test with fixed compression

Possible modifications

- Modification of the plasticity criterion



[Mouzannar2016]

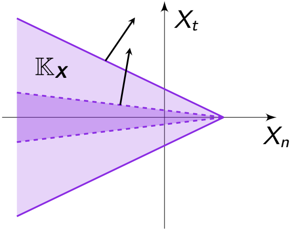


[Ghasempour *et al.*2017]

- Evolution of the plasticity criterion with damage

$$f_{\boldsymbol{X}}(\boldsymbol{X}) = \|\boldsymbol{X_t}\| + \mu X_n - c \leq 0$$

$$\mu \to 0 \qquad \text{with } \mu(\alpha) \text{ or } \mu(Y)$$



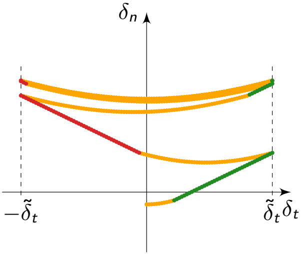
- Addition of hyperelasticity

$$K_n(\delta_n), K_n(\boldsymbol{\delta_t}), K_n(\delta_n, \boldsymbol{\delta_t}) \qquad K_t(\delta_n), K_t(\boldsymbol{\delta_t}), K_t(\delta_n, \boldsymbol{\delta_t})$$

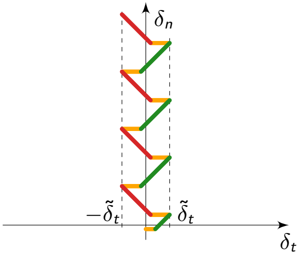
- Relation between $\boldsymbol{\sigma}$ and $\boldsymbol{X}$:

$$\begin{cases} \sigma_n = X_n + A_n(\alpha)p_n - \beta_n(X_n + A_n(\alpha)p_n)^2 - \beta_t\|\boldsymbol{X_t} + A_t(\alpha)\boldsymbol{p_t}\|^2 \\ \boldsymbol{\sigma_t} = \boldsymbol{X_t} + A_t(\alpha)\boldsymbol{p_t} \end{cases}$$

- $\mathbb{K}_{\boldsymbol{\sigma}}$ is fixed during hyperelastic loadings
- The asymptotic dilatancy is related to the normal to $\mathbb{K}_{\boldsymbol{\sigma}}$
- Cyclic shear loading (asymptotic behavior, i.e., without damage):



With hyperelasticity



Without hyperelasticity