An equilibrated a posteriori error analysis for frictional contact problems

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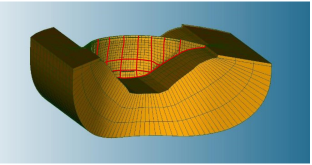
Motivation - Industrial context

- Finite element numerical simulations to study large hydraulic structures and evaluate their safety
- Gleno (Italy, 1923), Malpasset (France, 1959)
- Nonlinearity at the interface level
- Concrete dams show different interface zones:
  - concrete-rock contact in the foundation
  - joints between the blocks of the dam
  - joints in concrete
  - ...
- Need for accurate simulations



Gleno

Malpasset

A posteriori error estimate

The error between the exact solution $\boldsymbol{u}$ and the approximate one $\boldsymbol{u}_h$ is measured with $|||\boldsymbol{u} - \boldsymbol{u}_h|||$, where $|||\cdot|||$ is a suitable norm.

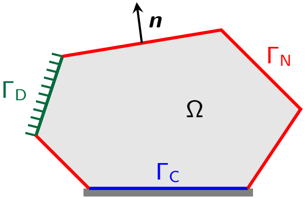
$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq \left(\sum_{T \in \mathcal{T}_h} \eta_T(\boldsymbol{u}_h)^2 \right)^{1/2}$$

Properties of a good a posteriori error estimate:

- Guaranteed error control
- Identification and separation of different components of the error
- Local efficiency ($\eta_T(\boldsymbol{u}_h) \leq C\,|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\mathcal{T}_T}$ for any element T)
- Error localization
- Adaptive mesh refinement (with some stopping criteria)

$\rightarrow$ A posteriori analysis via equilibrated stress/flux reconstruction

Unilateral contact problem with friction



Strong formulation

$$\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega, \tag{1a}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{E}\,\varepsilon(\boldsymbol{u}) \quad \text{in } \Omega, \tag{1b}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{D}}, \tag{1c}$$

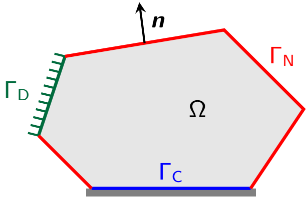
$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g}_{\mathrm{N}} \quad \text{on } \Gamma_{\mathrm{N}}, \tag{1d}$$

$$u^n \leq 0, \;\; \sigma^n(\boldsymbol{u}) \leq 0, \;\; \sigma^n(\boldsymbol{u})\,u^n = 0 \quad \text{on } \Gamma_{\mathrm{C}}, \tag{1e}$$

$$\begin{cases} |\boldsymbol{\sigma}^t(\boldsymbol{u})| \leq S(\boldsymbol{u}) & \text{if } \boldsymbol{u}^t = \boldsymbol{0} \\ \boldsymbol{\sigma}^t(\boldsymbol{u}) = -S(\boldsymbol{u})\dfrac{\boldsymbol{u}^t}{|\boldsymbol{u}^t|} & \text{otherwise} \end{cases} \quad \text{on } \Gamma_{\mathrm{C}}. \tag{1f}$$

- $\boldsymbol{u}\colon \Omega(\subseteq \mathbb{R}^d) \to \mathbb{R}^d$, $d \in \{2,3\}$ is the unknown displacement
- $\varepsilon(\boldsymbol{u})$ is the strain tensor, and $\boldsymbol{\sigma}(\boldsymbol{u}) = \lambda \mathrm{tr}\varepsilon(\boldsymbol{u})\boldsymbol{I}_d + 2\mu\varepsilon(\boldsymbol{u})$ is the stress tensor
- $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g}_{\mathrm{N}} \in \boldsymbol{L}^2(\Gamma_{\mathrm{N}})$ are volume and surface forces, respectively
- $\boldsymbol{u} = u^n\boldsymbol{n} + \boldsymbol{u}^t$ and $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma^n(\boldsymbol{u})\boldsymbol{n} + \boldsymbol{\sigma}^t(\boldsymbol{u})$ on Γ_{C}
- $S(\boldsymbol{u})$ fixes the friction conditions; $S(\boldsymbol{u}) = s \in L^2(\Gamma_{\mathrm{C}})$, $s \geq 0$, for the Tresca friction model, and $S(\boldsymbol{u}) = -\mu_{\mathrm{Coul}}\,\sigma^n(\boldsymbol{u})$ for the Coulomb one

Unilateral contact problem with friction



Strong formulation

$$\mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \mathbf{0} \quad \text{in } \Omega, \tag{1a}$$

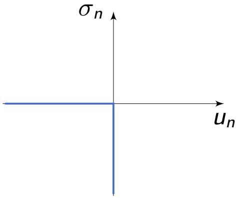
$$\boldsymbol{\sigma}(\boldsymbol{u}) = \mathbb{E}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{1b}$$

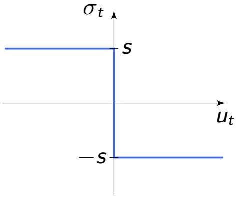
$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_D, \tag{1c}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g}_N \quad \text{on } \Gamma_N, \tag{1d}$$

$$u^n \leq 0,\ \sigma^n(\boldsymbol{u}) \leq 0,\ \sigma^n(\boldsymbol{u})\, u^n = 0 \quad \text{on } \Gamma_C, \tag{1e}$$

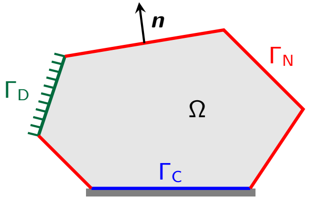
$$\begin{cases} |\boldsymbol{\sigma}^t(\boldsymbol{u})| \leq S(\boldsymbol{u}) & \text{if } \boldsymbol{u}^t = \mathbf{0} \\ \boldsymbol{\sigma}^t(\boldsymbol{u}) = -S(\boldsymbol{u}) \dfrac{\boldsymbol{u}^t}{|\boldsymbol{u}^t|} & \text{otherwise} \end{cases} \quad \text{on } \Gamma_C. \tag{1f}$$





- Tresca friction: $S(\boldsymbol{u}) = s \in L^2(\Gamma_C)$, $s \geq 0$
- Coulomb friction: $S(\boldsymbol{u}) = -\mu_{\mathrm{Coul}}\, \sigma^n(\boldsymbol{u})$

Unilateral contact problem with friction



Strong formulation

$$\begin{aligned}
\mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} &= \boldsymbol{0} && \text{in } \Omega, && \text{(1a)}\\
\boldsymbol{\sigma}(\boldsymbol{u}) &= \mathbb{E}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) && \text{in } \Omega, && \text{(1b)}\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_{\mathrm{D}}, && \text{(1c)}\\
\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{g}_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}, && \text{(1d)}\\
u^n \leq 0,\ \sigma^n(\boldsymbol{u}) \leq 0,\ \sigma^n(\boldsymbol{u})\, u^n &= 0 && \text{on } \Gamma_{\mathrm{C}}, && \text{(1e)}\\
\begin{cases} |\boldsymbol{\sigma}^t(\boldsymbol{u})| \leq S(\boldsymbol{u}) & \text{if } \boldsymbol{u}^t = \boldsymbol{0} \\ \boldsymbol{\sigma}^t(\boldsymbol{u}) = -S(\boldsymbol{u}) \dfrac{\boldsymbol{u}^t}{|\boldsymbol{u}^t|} & \text{otherwise} \end{cases} & && \text{on } \Gamma_{\mathrm{C}}. && \text{(1f)}
\end{aligned}$$

$$\boldsymbol{H}^1_{\mathrm{D}}(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \ :\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{D}} \right\}$$

$$\boldsymbol{K} := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1_{\mathrm{D}}(\Omega) \ :\ v^n \leq 0 \text{ on } \Gamma_{\mathrm{C}} \right\}$$

Weak formulation

Find $\boldsymbol{u} \in \boldsymbol{K}$ such that

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{u})\big) + (S(\boldsymbol{u}), |\boldsymbol{v}^t|)_{\Gamma_{\mathrm{C}}} - (S(\boldsymbol{u}), |\boldsymbol{u}^t|)_{\Gamma_{\mathrm{C}}} \geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}) + (\boldsymbol{g}_{\mathrm{N}}, \boldsymbol{v}-\boldsymbol{u})_{\Gamma_{\mathrm{N}}} \qquad \forall \boldsymbol{v} \in \boldsymbol{K} \qquad (2)$$

Unilateral contact problem - Numerical approach

- $\mathcal{T}_h$ be a triangulation of Ω and $\boldsymbol{V}_h := \boldsymbol{H}^1_{\mathrm{D}}(\Omega) \cap \boldsymbol{\mathcal{P}}^p(\mathcal{T}_h)$, $p \geq 1$.
- $[\cdot]_{\mathbb{R}-}$ is the projection operator on the half-line of negative real numbers
- $[\cdot]_\alpha$ is the projection operator on the $(d-1)$-dimensional ball $B(\mathbf{0}, \alpha)$

The contact boundary conditions (1e) and (1f) can be rewritten as

$$\sigma^n(\boldsymbol{u}) = [\sigma^n(\boldsymbol{u}) - \gamma u^n]_{\mathbb{R}-} =: \left[P^n_\gamma(\boldsymbol{u})\right]_{\mathbb{R}-} \tag{3a}$$

$$\boldsymbol{\sigma}^{\boldsymbol{t}}(\boldsymbol{u}) = \left[\boldsymbol{\sigma}^{\boldsymbol{t}}(\boldsymbol{u}) - \gamma \boldsymbol{u}^{\boldsymbol{t}}\right]_{S(\boldsymbol{u})} =: \left[\boldsymbol{P}^{\boldsymbol{t}}_\gamma(\boldsymbol{u})\right]_{S(\boldsymbol{u})} \tag{3b}$$

Nitsche-based method

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}_h), \varepsilon(\boldsymbol{v}_h)\big) - \left(\left[P^n_\gamma(\boldsymbol{u}_h)\right]_{\mathbb{R}-}, v^n_h\right)_{\Gamma_{\mathrm{C}}} - \left(\left[\boldsymbol{P}^{\boldsymbol{t}}_\gamma(\boldsymbol{u}_h)\right]_{S(\boldsymbol{u}_h)}, \boldsymbol{v}^{\boldsymbol{t}}_h\right)_{\Gamma_{\mathrm{C}}} =$$

$$= (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{g}_{\mathrm{N}}, \boldsymbol{v}_h)_{\Gamma_{\mathrm{N}}} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

A posteriori analysis - Measure of the error

At the k-th iteration of the Newton algorithm, we define the residual operator $\mathcal{R}(\boldsymbol{u}_h^k) \in (\boldsymbol{H}_D^1(\Omega))^*$ by

$$\langle \mathcal{R}(\boldsymbol{u}_h^k), \boldsymbol{v} \rangle := (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{g}_{\mathrm{N}}, \boldsymbol{v})_{\Gamma_{\mathrm{N}}} - \left(\boldsymbol{\sigma}(\boldsymbol{u}_h^k), \varepsilon(\boldsymbol{v})\right) + \left(\left[P_\gamma^n(\boldsymbol{u}_h^k) \right]_{\mathbb{R}^-}, v^n \right)_{\Gamma_C} + \left(\left[\boldsymbol{P}_\gamma^{\boldsymbol{t}}(\boldsymbol{u}_h^k) \right]_{S(\boldsymbol{u}_h^k)}, \boldsymbol{v}^{\boldsymbol{t}} \right)_{\Gamma_C} \quad (4)$$

for all $\boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega)$. Then, the error between $\boldsymbol{u}$ and $\boldsymbol{u}_h^k$ is measured by the dual norm

$$\left\| \mathcal{R}(\boldsymbol{u}_h^k) \right\|_* := \sup_{\substack{\boldsymbol{v} \in \boldsymbol{H}_{\mathrm{D}}^1(\Omega), \\ \|\boldsymbol{v}\|_{\mathrm{C},h}=1}} \langle \mathcal{R}(\boldsymbol{u}_h^k), \boldsymbol{v} \rangle \quad (5)$$

where $\|\cdot\|_{\mathrm{C},h}$ is a norm which takes into account the contact boundary part:

$$\|\boldsymbol{v}\|_{\mathrm{C},h}^2 := \|\boldsymbol{\nabla v}\|^2 + \sum_{F \in \mathcal{F}_h^{\mathrm{C}}} \frac{1}{h_F} \|\boldsymbol{v}\|_F^2 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{\mathrm{D}}^1(\Omega). \quad (6)$$

$\Rightarrow$ Comparison between the residual dual norm and the energy norm

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{en}}^2 = (\boldsymbol{\sigma}(\boldsymbol{u} - \boldsymbol{u}_h), \varepsilon(\boldsymbol{u} - \boldsymbol{u}_h))$$

A posteriori analysis - Stress reconstruction

In general,

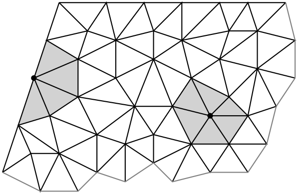
$$\boldsymbol{u}_h^k \in \boldsymbol{H}^1_{\mathrm{D}}(\Omega) \qquad \text{but} \qquad \begin{cases} \boldsymbol{\sigma}(\boldsymbol{u}_h^k) \notin \mathbb{H}(\mathrm{div},\Omega) \\ \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}_h^k) \neq -\boldsymbol{f} \\ \boldsymbol{\sigma}(\boldsymbol{u}_h^k)\boldsymbol{n} \neq \boldsymbol{g}_{\mathrm{N}} \text{ on } \Gamma_{\mathrm{N}} \end{cases}$$

where $\mathbb{H}(\mathrm{div},\Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \mid \mathbf{div}\,\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)\}$.

Stress reconstruction: $\begin{cases} \boldsymbol{\sigma}_h^k \in \mathbb{H}(\mathrm{div},\Omega) \\ (\mathbf{div}\,\boldsymbol{\sigma}_h^k + \boldsymbol{f}, \boldsymbol{v}_T)_T = 0 & \forall \boldsymbol{v}_T \in \boldsymbol{\mathcal{P}}^0(T), \forall T \in \mathcal{T}_h \\ (\boldsymbol{\sigma}_h^k\boldsymbol{n}, \boldsymbol{v}_F)_F = (\boldsymbol{g}_{\mathrm{N}}, \boldsymbol{v}_F)_F & \forall \boldsymbol{v}_F \in \boldsymbol{\mathcal{P}}^0(F), \forall F \in \mathcal{F}_h^{\mathrm{N}} \end{cases}$

$$\boldsymbol{\sigma}_h^k = \underbrace{\boldsymbol{\sigma}_{h,\mathrm{dis}}^k}_{\text{discretization}} + \underbrace{\boldsymbol{\sigma}_{h,\mathrm{lin}}^k}_{\text{linearization}}$$

Local problems defined on patches using Arnold–Falk–Winther FE space.



$\Rightarrow$ Equilibrated, H-div conforming and weakly symmetric tensor $\boldsymbol{\sigma}_h^k$

A posteriori analysis

Theorem (A posteriori error estimate)

$$\left\|\mathcal{R}(\boldsymbol{u}_h^k)\right\|_* \leq \left(\sum_{T\in\mathcal{T}_h}\left((\eta^k_{a,T})^2 + (\eta^k_{b,T})^2\right)\right)^{1/2}$$

where

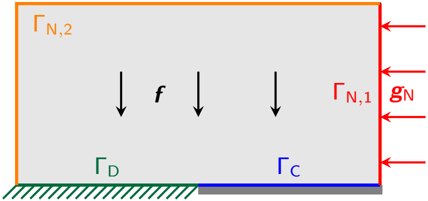
$$\eta^k_{a,T} := \eta^k_{\mathrm{osc},T} + \eta^k_{\mathrm{str},T} + \eta^k_{\mathrm{Neu},T} + \eta^k_{\mathrm{lin1},T},$$
$$\eta^k_{a,T} := \eta^k_{\mathrm{cnt},T} + \eta^k_{\mathrm{frc},T} + \eta^k_{\mathrm{lin2n},T} + \eta^k_{\mathrm{lin2t},T}.$$

Adaptive algorithm

- Only the elements where $\eta_{\mathrm{tot},T} := \left((\eta^k_{a,T})^2 + (\eta^k_{b,T})^2\right)^{1/2}$ is high are refined.
- The number of Newton iterations and the value of δ can be fixed automatically by the algorithm using a stopping criterion:

$$\eta^k_{\mathrm{lin1}} + \eta^k_{\mathrm{lin2n}} + \eta^k_{\mathrm{lin2t}} \leq \gamma_{\mathrm{lin}}(\eta^k_{\mathrm{osc}} + \eta^k_{\mathrm{str}} + \eta^k_{\mathrm{Neu}} + \eta^k_{\mathrm{cnt}} + \eta^k_{\mathrm{frc}}). \tag{7}$$

Numerical results - Tresca friction



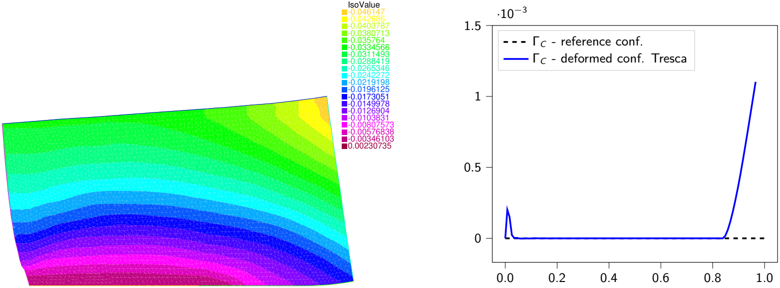
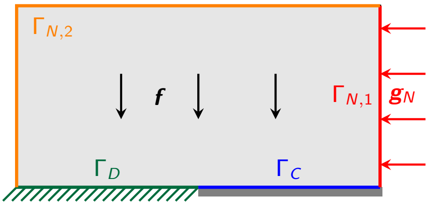


Figure: Horizontal displacement (*left*, amplification factor = 5) and profile of Γ_C (*right*) in the deformed domain.

Adaptive mesh refinement



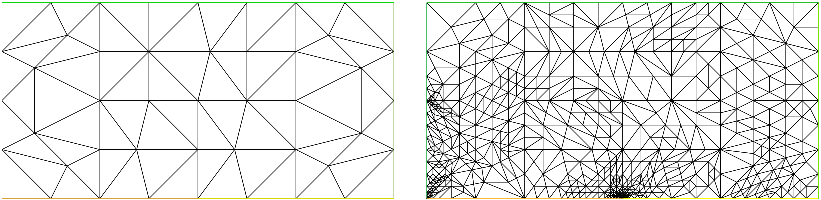
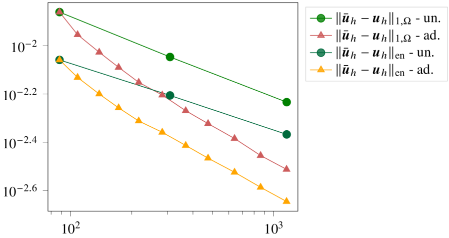
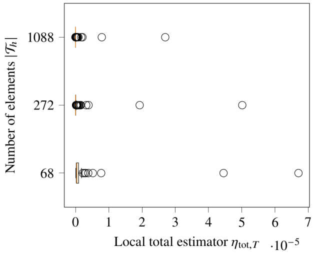


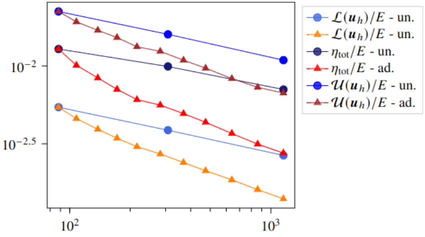
Figure: Initial mesh (*left*) and adaptively refined mesh after 10 steps (*right*).

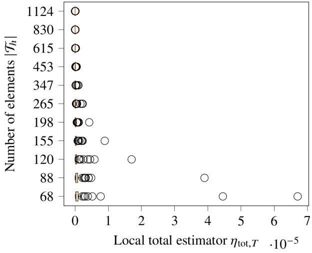
Adaptive VS Uniform refinement

$$\|\boldsymbol{v}\|_{\mathrm{en}}^2 := (\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))$$









Stopping criteria

	Initial	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
N_{lin}	3	3	3	3	4	4	4	5	5	5	5

Table: Number of regularization iterations N_{reg} and Newton iterations N_{lin} at each refinement step of the adaptive algorithm with the stopping criteria (9) and (10).

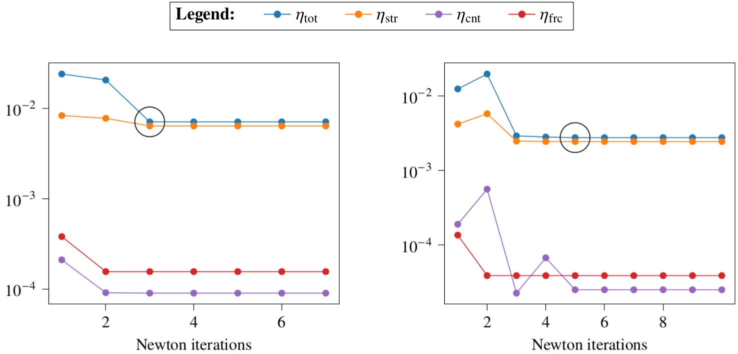
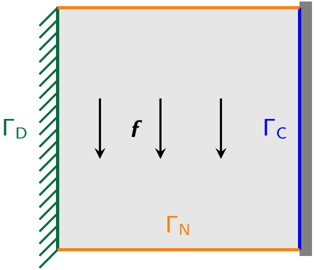
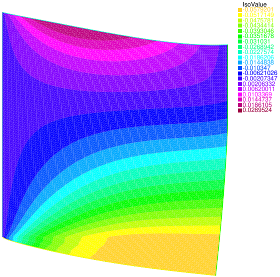


Figure: 3^{rd} (*left*) and 10^{th} (*right*) adaptively refined mesh

Numerical results - Coulomb friction





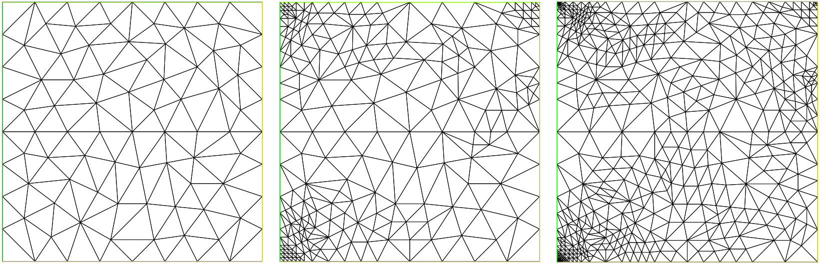
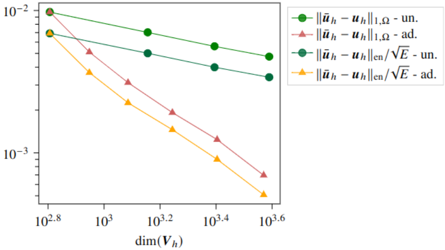
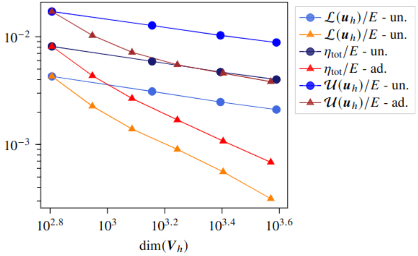


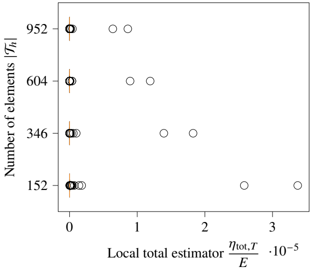
Figure: Initial mesh (*left*) and adaptively refined mesh after 3 steps (*middle*) and 5 steps (*right*).

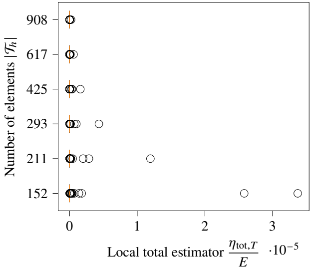
Adaptive VS Uniform refinement

$$\|\boldsymbol{v}\|_{\text{en}}^2 := (\boldsymbol{\sigma}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v}))$$









Conclusions:

- Nitsche-based method applied to the unilateral contact problem with friction, including both Tresca and Coulomb friction.
- "Generalized" Newton method.
- A posteriori estimate of the error measured with a dual norm for the frictional contact problem via stress reconstruction.
- We distinguish the different error components and we propose an adaptive algorithm with stopping criterion.
- Better asymptotic convergence with adaptive refinement.

Perspectives:

- Extension to the contact between two bodiess
- Extension to contact problem with cohesive forces
- Industrial application on hydraulic structures

Thank you for your attention!

References - A posteriori error analysis (selection)

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