A Posteriori Error Estimation via Equilibrated Stress Reconstruction for Unilateral Contact Problems

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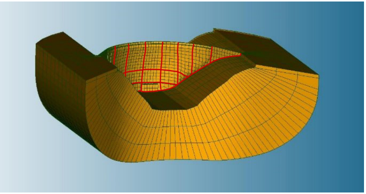
Motivation - Industrial context

- Engineering teams use finite element numerical simulations to study large hydraulic structures and evaluate their safety.
- Gleno (Italy, 1923), Malpasset (France, 1959)
- Concrete dams show different interface zones:
  - concrete-rock contact in the foundation
  - joints between the blocks of the dam
  - joints in concrete
  - ...
- Need for accurate simulations



Gleno

Malpasset

A posteriori estimation background

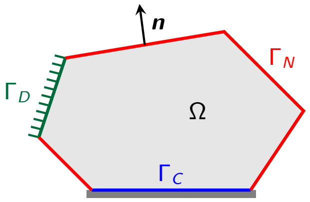
A posteriori error estimation:

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq \left(\sum_{T \in \mathcal{T}_h} \eta_T(\boldsymbol{u}_h)^2 \right)^{1/2}$$

where $\boldsymbol{u}$ is the exact solution of the considered problem, and $\boldsymbol{u}_h$ is an approximate solution. The error between the exact solution and the approximate one is measured with $|||\boldsymbol{u} - \boldsymbol{u}_h|||$, where $|||\cdot|||$ is some norm.

- Error control
- Local and global efficiency ($\eta_T(\boldsymbol{u}_h) \leq C\,|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\mathcal{T}_T}$ for every element T)
- Error localization
- Identification and separation of different components of the error
- Adaptive mesh refinement

Unilateral contact problem

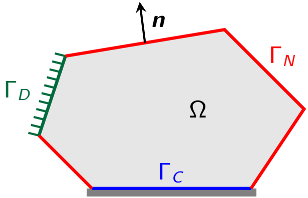


Strong formulation

$$\begin{aligned}
\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} &= \boldsymbol{0} && \text{in } \Omega, && \text{(1a)}\\
\boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{A} : \varepsilon(\boldsymbol{u}) && \text{in } \Omega, && \text{(1b)}\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_D, && \text{(1c)}\\
\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{g}_N && \text{on } \Gamma_N, && \text{(1d)}\\
u^n \leq 0,\ \sigma^n(\boldsymbol{u}) \leq 0,\ \sigma^n(\boldsymbol{u})\, u^n &= 0 && \text{on } \Gamma_C, && \text{(1e)}\\
\boldsymbol{\sigma}^{\boldsymbol{t}}(\boldsymbol{u}) &= \boldsymbol{0} && \text{on } \Gamma_C, && \text{(1f)}
\end{aligned}$$

- $\boldsymbol{u}\colon \Omega(\subseteq \mathbb{R}^d) \to \mathbb{R}^d$, $d \in \{2,3\}$ is the unknown displacement
- $\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{ij}$, where $\varepsilon_{ij}(\boldsymbol{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, is the strain tensor
- $\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{A} : \varepsilon(\boldsymbol{u}) := \lambda \mathrm{tr}\varepsilon(\boldsymbol{u})\boldsymbol{I}_d + 2\mu\varepsilon(\boldsymbol{u})$ is the elasticity stress tensor
- $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g}_N \in \boldsymbol{L}^2(\Gamma_N)$ are volume and surface forces, respectively
- $\boldsymbol{u} = u^n\boldsymbol{n} + \boldsymbol{u}^{\boldsymbol{t}}$ and $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma^n(\boldsymbol{u})\boldsymbol{n} + \boldsymbol{\sigma}^{\boldsymbol{t}}(\boldsymbol{u})$ on Γ_C

Unilateral contact problem



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\end{aligned}$$

$$\boldsymbol{H}^1_D(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \ : \ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D \right\}$$
$$\boldsymbol{K} := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega) \ : \ v^n \leq 0 \text{ on } \Gamma_C \right\}$$

Weak formulation

Find $\boldsymbol{u} \in \boldsymbol{K}$ such that

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}), \varepsilon(\boldsymbol{v}-\boldsymbol{u})\big) \geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u}) + (\boldsymbol{g}_N, \boldsymbol{v}-\boldsymbol{u})_{\Gamma_N} \qquad \forall \boldsymbol{v} \in \boldsymbol{K}. \qquad (2)$$

Unilateral contact problem - Numerical approach

Let $\mathcal{T}_h$ be a triangulation of Ω, $\boldsymbol{V}_h := \boldsymbol{H}^1_D(\Omega) \cap \boldsymbol{\mathcal{P}}^p(\mathcal{T}_h)$, $p \geq 1$, and $[\cdot]_{\mathbb{R}^-}$ the projection on the half-line of negative real numbers $\mathbb{R}^-$.

Nitsche-based method

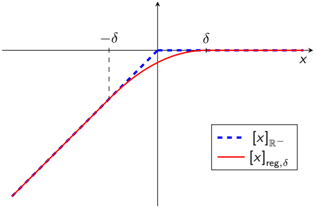
Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) - \left(\left[P^n_{1,\gamma}(\boldsymbol{u}_h)\right]_{\mathbb{R}^-}, v^n_h\right)_{\Gamma_C} = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{g}_N, \boldsymbol{v}_h)_{\Gamma_N} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

where $P^n_{1,\gamma}(\boldsymbol{u}_h) := \sigma^n(\boldsymbol{u}_h) - \gamma u^n_h$.

In order to solve this nonlinear problem

1. we regularize the projection operator $[\cdot]_{\mathbb{R}^-}$ with $[\cdot]_{\text{reg},\delta}$,
2. we use Netwon method.



A posteriori analysis

At the k-th iteration of the Newton algorithm, the error between $\boldsymbol{u}$ and $\boldsymbol{u}_h^k$ is measured by

$$\left\|\mathcal{R}(\boldsymbol{u}_h^k)\right\|_{(\boldsymbol{H}_D^1(\Omega))^*} := \sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}_D^1(\Omega),\\ \|\boldsymbol{v}\|_{C,h}=1}} \langle \mathcal{R}(\boldsymbol{u}_h^k), \boldsymbol{v}\rangle_{(\boldsymbol{H}_D^1(\Omega))^*,\boldsymbol{H}_D^1(\Omega)}$$

where $\mathcal{R}(\boldsymbol{u}_h^k)$ is the residual of $\boldsymbol{u}_h^k$, and $\|\cdot\|_{C,h}$ is a norm which takes into account the boundary contact part.

$$\boldsymbol{u}_h^k \in \boldsymbol{H}_D^1(\Omega) \qquad \text{but} \qquad \boldsymbol{\sigma}(\boldsymbol{u}_h^k) \notin \mathbb{H}(\mathrm{div},\Omega)$$

Stress reconstruction: $\boldsymbol{\sigma}_h^k \in \mathbb{H}(\mathrm{div},\Omega)$

$$\boldsymbol{\sigma}_h^k = \boldsymbol{\sigma}_{h,1}^k + \underbrace{\boldsymbol{\sigma}_{h,2}^k}_{\text{regularization}} + \underbrace{\boldsymbol{\sigma}_{h,3}^k}_{\text{linearization}}$$

Each term is obtained through local problems defined on patches around the vertices of the mesh.

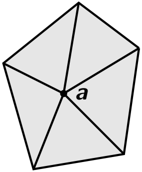


Figure: Patch around a node

A posteriori analysis

Theorem (A posteriori error estimate)

$$\left\|\mathcal{R}(\boldsymbol{u}_h^k)\right\|_{(\boldsymbol{H}_D^1(\Omega))^*} \leq \left(\sum_{T\in\mathcal{T}_h} (\eta_{\mathrm{tot},T}^k)^2\right)^{1/2}$$

where

$$\eta_{\mathrm{tot},T}^k := \eta_{\mathrm{osc},T}^k + \eta_{\mathrm{str},T}^k + \eta_{\mathrm{Neu},T}^k + \eta_{\mathrm{cnt},T}^k + \eta_{\mathrm{reg},T}^k + \eta_{\mathrm{lin},T}^k.$$

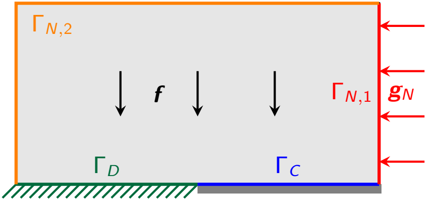
Adaptive algorithm

- Only the elements where $\eta_{\mathrm{tot},T}$ is higher are refined.
- The number of Newton iterations and the value of δ can be fixed automatically by the algorithm using some stopping criteria:

$$\eta_{\mathrm{reg}}^k \leq \gamma_{\mathrm{reg}}(\eta_{\mathrm{osc}}^k + \eta_{\mathrm{str}}^k + \eta_{\mathrm{Neu}}^k + \eta_{\mathrm{cnt}}^k + \eta_{\mathrm{lin}}^k), \tag{3}$$

$$\eta_{\mathrm{lin}}^k \leq \gamma_{\mathrm{lin}}(\eta_{\mathrm{osc}}^k + \eta_{\mathrm{str}}^k + \eta_{\mathrm{Neu}}^k + \eta_{\mathrm{cnt}}^k). \tag{4}$$

Numerical results



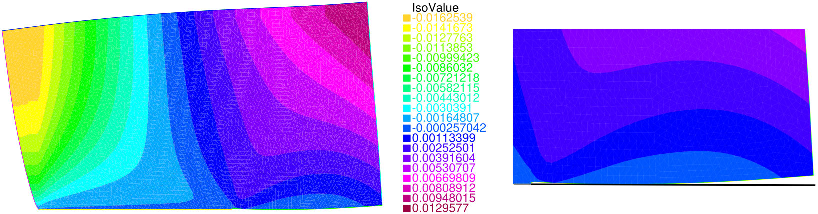
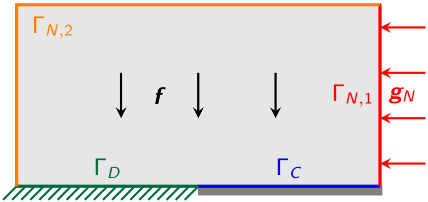
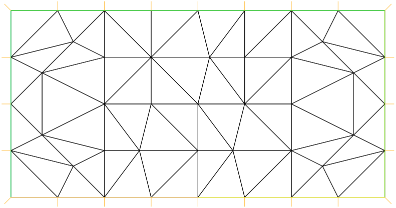
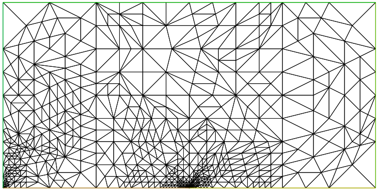


Figure: Vertical displacement in the deformed domain (amplification factor = 5): whole domain (left) and zoom near the contact boundary (right).

Adaptive mesh refinement

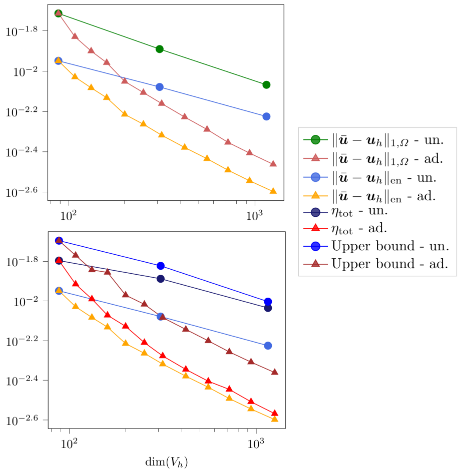


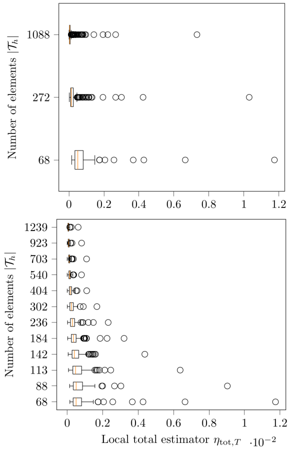




Adaptive VS Uniform refinement

$$\|\boldsymbol{v}\|_{\text{en}} := (\boldsymbol{\sigma}(\boldsymbol{v}), \varepsilon(\boldsymbol{v}))$$





Conclusions:

- Nitsche-based method applied to the unilateral contact problem without friction.
- Regularization and linearization steps.
- A posteriori estimate of the error measured with a dual norm.
- We distinguish the different error components.
- Better asymptotic convergence with adaptive refinement.

Perspectives:

- Extension to the unilateral problem with friction and bilateral problem.
- Extension to contact problem with cohesive forces.
- Industrial application on hydraulic structures.

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