A Posteriori Error Estimation via Equilibrated Stress Reconstruction for Unilateral Contact Problems

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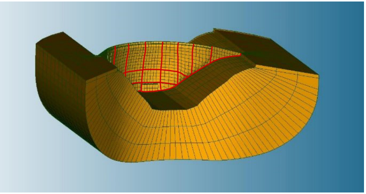
Motivation - Industrial context

- Engineering teams use finite element numerical simulations to study large hydraulic structures and evaluate their safety.
- Gleno (Italy, 1923), Malpasset (France, 1959)
- Concrete dams show different interface zones:
  - concrete-rock contact in the foundation
  - joints between the blocks of the dam
  - joints in concrete
  - ...
- Need for accurate simulations



Gleno

Malpasset

Finite element approximation background

We consider a problem on a domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$ which is expressed by some Partial Differential Equations.

Strong formulation	→	Weak formulation	→	Approximated problem
PDE, Ω, $\boldsymbol{u} \in \mathcal{C}^p(\Omega)$		$\mathcal{P}$, Ω, $\boldsymbol{u} \in \boldsymbol{V}$		$\mathcal{P}_h$, $\mathcal{T}_h$, $\boldsymbol{u}_h \in \boldsymbol{V}_h$

- $\boldsymbol{V}$ is a space of function infinite-dimensional, $\boldsymbol{V}_h$ is a finite-dimensional approximation of $\boldsymbol{V}$
- $\boldsymbol{u}$ is the *exact solution*, $\boldsymbol{u}_h$ is an *approximated solution* found using a numerical method
- $\mathcal{T}_h$ is a *spatial mesh*, i.e., a partition of Ω

An example: Poisson problem in one-dimensional space

$\Omega = (a, b) \subset \mathbb{R}$, $u' := \dfrac{\mathrm{d}u}{\mathrm{d}x}$

Strong formulation: Find $u \in \mathcal{C}^2(\Omega)$ such that

$$u'' + f = 0 \qquad \text{in } \Omega \tag{1a}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{1b}$$

Weak formulation: Find $u \in H_0^1(\Omega)$ such that

$$(u', v') = (f, v) \qquad v \in H_0^1(\Omega), \tag{2}$$

where $H_0^1(\Omega) := \{v \in H^1(\Omega) | v = 0 \text{ on } \partial\Omega\}$.

Approximated problem: Find $u_h \in V_h$ such that

$$(u_h', v_h') = (f, v_h) \qquad v_h \in V_h, \tag{3}$$

where $V_h = \{v_h \in \mathcal{C}^0(\overline{\Omega}) | v_h|_T \in \mathcal{P}^p(T) \ \forall T \in \mathcal{T}_h\}$.

A posteriori estimation background

The error between the exact solution and the approximate solution is measured with $|||\boldsymbol{u} - \boldsymbol{u}_h|||$, where $|||\cdot|||$ is some norm.

***A priori* error estimate:**

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq C(\boldsymbol{u})h^k$$

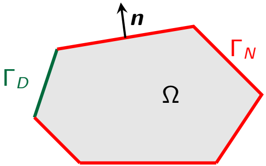
***A posteriori* error estimate:**

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| \leq \left(\sum_{T \in \mathcal{T}_h} \eta_T(\boldsymbol{u}_h)^2\right)^{1/2}$$

Features of a good *a posteriori* error estimate:

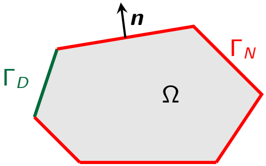
- Error control
- Local efficiency ($\eta_T(\boldsymbol{u}_h) \leq C\, |||\boldsymbol{u} - \boldsymbol{u}_h|||_{\mathcal{T}_T}$ for every element T)
- Error localization
- Identification and separation of different components of the error
- Adaptive mesh refinement

Elasto-static problem background



- Small deformation hypothesis
- Ω is the domain which represents an elastic body (reference configuration)
- $\boldsymbol{u}\colon \Omega(\subseteq \mathbb{R}^d) \to \mathbb{R}^d$, $d \in \{2,3\}$ is the unknown displacement
- $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{ij}$, where $\varepsilon_{ij}(\boldsymbol{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, is the strain tensor
- $\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{A} : \varepsilon(\boldsymbol{u}) := \lambda \mathrm{tr}\varepsilon(\boldsymbol{u})\boldsymbol{I}_d + 2\mu\varepsilon(\boldsymbol{u})$ is the elasticity stress tensor

Elasto-static problem background



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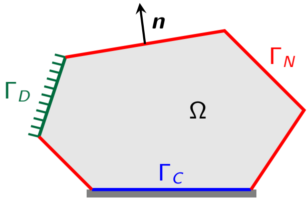
Elasto-static problem

$$\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega, \tag{4a}$$
$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on } \Gamma_D, \tag{4b}$$
$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g}_N \quad \text{on } \Gamma_N \tag{4c}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \quad \text{in } \Omega, \tag{5a}$$
$$u_i = u_{D,i} \quad \text{on } \Gamma_D, \tag{5b}$$
$$\sigma_{ij} n_j = g_{N,i} \quad \text{on } \Gamma_N \tag{5c}$$

Unilateral contact problem



Strong formulation

$$\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega, \tag{6a}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{A} : \varepsilon(\boldsymbol{u}) \quad \text{in } \Omega, \tag{6b}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{6c}$$

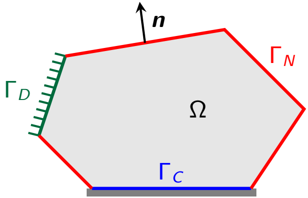
$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g}_N \quad \text{on } \Gamma_N, \tag{6d}$$

$$u^n \leq 0, \ \sigma^n(\boldsymbol{u}) \leq 0, \ \sigma^n(\boldsymbol{u})\, u^n = 0 \quad \text{on } \Gamma_C, \tag{6e}$$

$$\boldsymbol{\sigma}^t(\boldsymbol{u}) = \boldsymbol{0} \quad \text{on } \Gamma_C \tag{6f}$$

- $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ represents volume forces
- $\boldsymbol{g}_N \in \boldsymbol{L}^2(\Gamma_N)$ represents surface forces
- $\boldsymbol{u} = u^n \boldsymbol{n} + \boldsymbol{u}^t$ on Γ_C
- $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \sigma^n(\boldsymbol{u})\boldsymbol{n} + \boldsymbol{\sigma}^t(\boldsymbol{u})$ on Γ_C

Unilateral contact problem



Strong formulation

$$\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = \boldsymbol{0} \quad \text{in } \Omega, \tag{6a}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{A} : \varepsilon(\boldsymbol{u}) \quad \text{in } \Omega, \tag{6b}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_D, \tag{6c}$$

$$\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g}_N \quad \text{on } \Gamma_N, \tag{6d}$$

$$u^n \leq 0, \ \sigma^n(\boldsymbol{u}) \leq 0, \ \sigma^n(\boldsymbol{u})\, u^n = 0 \quad \text{on } \Gamma_C, \tag{6e}$$

$$\boldsymbol{\sigma}^t(\boldsymbol{u}) = \boldsymbol{0} \quad \text{on } \Gamma_C \tag{6f}$$

$$\boldsymbol{H}^1_D(\Omega) := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \ : \ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D \right\}$$
$$\boldsymbol{K} := \left\{ \boldsymbol{v} \in \boldsymbol{H}^1_D(\Omega) \ : \ v^n \leq 0 \text{ on } \Gamma_C \right\}$$

Weak formulation

Find $\boldsymbol{u} \in \boldsymbol{K}$ such that

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}), \varepsilon(\boldsymbol{v} - \boldsymbol{u})\big) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{g}_N, \boldsymbol{v} - \boldsymbol{u})_{\Gamma_N} \qquad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{7}$$

Unilateral contact problem - Numerical approach

Let $\mathcal{T}_h$ be a triangulation of Ω, and $\boldsymbol{V}_h := \boldsymbol{H}^1_D(\Omega) \cap \boldsymbol{\mathcal{P}}^p(\mathcal{T}_h)$, $p \geq 1$. Moreover, we define $[\cdot]_{\mathbb{R}^-}$ as the projection on the half-line of negative real numbers $\mathbb{R}^-$, and the following operator

$$\begin{aligned} P_\gamma : \boldsymbol{V}_h &\to L^2(\Gamma_C) \\ \boldsymbol{v}_h &\mapsto \sigma^n(\boldsymbol{v}_h) - \gamma v_h^n. \end{aligned}$$

The contact boundary condition (6e) can be rewritten as

$$\sigma^n(\boldsymbol{u}) = [P_\gamma(\boldsymbol{u})]_{\mathbb{R}^-} . \tag{8}$$

Nitsche-based method

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\big) - \Big(\big[P_\gamma(\boldsymbol{u}_h)\big]_{\mathbb{R}^-}, v_h^n\Big)_{\Gamma_C} = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{g}_N, \boldsymbol{v}_h)_{\Gamma_N} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Unilateral contact problem - Numerical approach

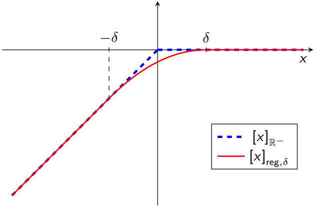
Nitsche-based method

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$\Big(\boldsymbol{\sigma}(\boldsymbol{u}_h), \varepsilon(\boldsymbol{v}_h)\Big) - \Big(\big[P_\gamma(\boldsymbol{u}_h)\big]_{\mathbb{R}-}, v_h^n\Big)_{\Gamma_C} = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{g}_N, \boldsymbol{v}_h)_{\Gamma_N} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

In order to solve this nonlinear problem

1. we regularize the projection operator $[\,\cdot\,]_{\mathbb{R}-}$ with $[\,\cdot\,]_{\text{reg},\delta}$,
2. we use Netwon method.



At each step $k \geq 1$ we have to solve the linear problem: Find $\boldsymbol{u}_h^k \in \boldsymbol{V}_h$ such that

$$\Big(\boldsymbol{\sigma}(\boldsymbol{u}_h^k), \varepsilon(\boldsymbol{v}_h)\Big) - \Big(P_{\text{lin}}^{k-1}(\boldsymbol{u}_h^k), v_h^n\Big)_{\Gamma_C} = (\boldsymbol{f}, \boldsymbol{v}_h) + (\boldsymbol{g}_N, \boldsymbol{v}_h)_{\Gamma_N} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \quad (9)$$

A posteriori analysis - Measure of the error

At the k-th iteration of the Newton algorithm, we define the residual operator $\mathcal{R}(\boldsymbol{u}_h^k) \in (\boldsymbol{H}_D^1(\Omega))^*$ by

$$\langle \mathcal{R}(\boldsymbol{u}_h^k), \boldsymbol{v} \rangle := (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{g}_N, \boldsymbol{v})_{\Gamma_N} - \left(\boldsymbol{\sigma}(\boldsymbol{u}_h^k), \varepsilon(\boldsymbol{v})\right) + \left(\left[P_\gamma(\boldsymbol{u}_h^k)\right]_{\mathbb{R}^-}, v^n \right)_{\Gamma_C} \quad (10)$$

for all $\boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega)$. Then, the error between $\boldsymbol{u}$ and $\boldsymbol{u}_h^k$ is measured by the dual norm

$$\left\| \mathcal{R}(\boldsymbol{u}_h^k) \right\|_{(\boldsymbol{H}_D^1(\Omega))^*} := \sup_{\substack{\boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega), \\ \|\boldsymbol{v}\|_{C,h}=1}} \langle \mathcal{R}(\boldsymbol{u}_h^k), \boldsymbol{v} \rangle \quad (11)$$

where $\|\cdot\|_{C,h}$ is a norm which takes into account the boundary contact part:

$$\|\boldsymbol{v}\|_{C,h}^2 := \|\boldsymbol{\nabla} \boldsymbol{v}\|^2 + \sum_{F \in \mathcal{F}_h^C} \frac{1}{h_F} \|\boldsymbol{v}\|_F^2 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_D^1(\Omega). \quad (12)$$

The example of Poisson problem

The error is measured by

$$\left\|(u-u_h)'\right\| = \sup_{\substack{v\in H_0^1(\Omega),\\ \|v'\|=1}} \left\{(f,v)-(u_h',v')\right\}, \qquad (13)$$

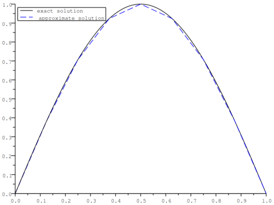
and we define the flux $\sigma(u) := u'$.

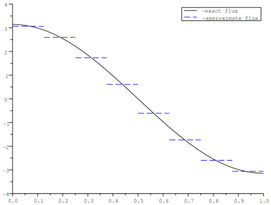
- Properties of the exact solution:

$$u \in H_0^1(\Omega) \qquad \text{and} \qquad \sigma(u) \in H^1(\Omega)$$

- Properties of the approximated solution

$$u_h \in H_0^1(\Omega) \qquad \text{but} \qquad \sigma(u_h) \notin H^1(\Omega) \text{ in general}$$





A posteriori analysis - Stress reconstruction

$$\boldsymbol{u}_h^k \in \boldsymbol{H}_D^1(\Omega) \qquad \text{but} \qquad \boldsymbol{\sigma}(\boldsymbol{u}_h^k) \notin \mathbb{H}(\text{div}, \Omega),$$

where $\mathbb{H}(\text{div}, \Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \mid \nabla \cdot \boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)\}$.

Stress reconstruction: $\boldsymbol{\sigma}_h^k \in \mathbb{H}(\text{div}, \Omega)$

$$\boldsymbol{\sigma}_h^k = \boldsymbol{\sigma}_{h,1}^k + \underbrace{\boldsymbol{\sigma}_{h,2}^k}_{\text{regularization}} + \underbrace{\boldsymbol{\sigma}_{h,3}^k}_{\text{linearization}}$$

$$\boldsymbol{\sigma}_{h,2}^k \to 0 \text{ as } \delta \to 0$$

$$\boldsymbol{\sigma}_{h,3}^k \to 0 \text{ as } k \to +\infty$$

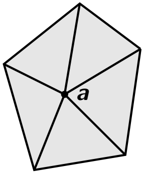


Figure: Patch around a node

Each term is obtained through local problems defined on patches around the vertices of the mesh using the Arnold-Falk-Winther mixed finite element space.

$\rightarrow$ Equilibrated, H-div conforming and weakly symmetric tensor $\boldsymbol{\sigma}_h^k$

A posteriori analysis

Theorem (A posteriori error estimate)

$$\left\|\mathcal{R}(\boldsymbol{u}_h^k)\right\|_{(\boldsymbol{H}_D^1(\Omega))^*} \leq \left(\sum_{T\in\mathcal{T}_h} (\eta_{\mathrm{tot},T}^k)^2\right)^{1/2}$$

where

$$\eta_{\mathrm{tot},T}^k := \eta_{\mathrm{osc},T}^k + \eta_{\mathrm{flux},T}^k + \eta_{\mathrm{Neu},T}^k + \eta_{\mathrm{disc},T}^k + \eta_{\mathrm{reg},T}^k + \eta_{\mathrm{lin},T}^k.$$

$$\eta_{\mathrm{osc},T}^k := \frac{h_T}{\pi}\left\|\boldsymbol{f} - \boldsymbol{\Pi}_T^{p-1}\boldsymbol{f}\right\|_T$$

$$\eta_{\mathrm{flux},T}^k := \|\boldsymbol{\sigma}_{h,1}^k - \boldsymbol{\sigma}(\boldsymbol{u}_h^k)\|_T$$

$$\eta_{\mathrm{Neu},T}^k := \sum_{F\in\mathcal{F}_T^C} C_{t,T,F}\, h_F^{1/2}\,\|\boldsymbol{g}_N - \boldsymbol{\Pi}_F^p \boldsymbol{g}_N\|_F$$

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$$\eta_{\text{disc},T}^k := \sum_{F\in\mathcal{F}_T^C} h_F^{1/2} \left\| \left[P_\gamma(\boldsymbol{u}_h^k)\right]_{\mathbb{R}-} - \Pi_F^p \left[P_\gamma(\boldsymbol{u}_h^k)\right]_{\mathbb{R}-} \right\|_F$$

$$\eta_{\text{reg},T}^k := \|\boldsymbol{\sigma}_{h,2}^k\|_T + \sum_{F\in\mathcal{F}_T^C} h_F^{1/2} \left\|\sigma_{h,2}^{k,n}\right\|_F$$

$$\eta_{\text{lin},T}^k := \|\boldsymbol{\sigma}_{h,3}^k\|_T + \sum_{F\in\mathcal{F}_T^C} h_F^{1/2} \left\|\sigma_{h,3}^{k,n}\right\|_F$$

A posteriori analysis

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$$\left\|\mathcal{R}(\boldsymbol{u}_h^k)\right\|_{(\boldsymbol{H}_D^1(\Omega))^*} \leq \left(\sum_{T\in\mathcal{T}_h} (\eta_{\mathrm{tot},T}^k)^2\right)^{1/2}$$

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Adaptive algorithm

- Only the element where $\eta_{\mathrm{tot},T}$ is high are refined.

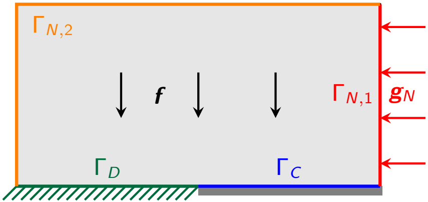
$$\eta_{\mathrm{reg},T}^k \to 0 \text{ as } \delta \to 0 \qquad \text{and} \qquad \eta_{\mathrm{lin},T}^k \to 0 \text{ as } k \to +\infty$$

- The number of Newton iterations and the value of δ can be fixed automatically by the algorithm using some stopping criteria:

$$\eta_{\mathrm{reg}}^k \leq \gamma_{\mathrm{reg}}(\eta_{\mathrm{osc}}^k + \eta_{\mathrm{flux}}^k + \eta_{\mathrm{Neu}}^k + \eta_{\mathrm{disc}}^k + \eta_{\mathrm{lin}}^k), \tag{14}$$

$$\eta_{\mathrm{lin}}^k \leq \gamma_{\mathrm{lin}}(\eta_{\mathrm{osc}}^k + \eta_{\mathrm{flux}}^k + \eta_{\mathrm{Neu}}^k + \eta_{\mathrm{disc}}^k). \tag{15}$$

Numerical results



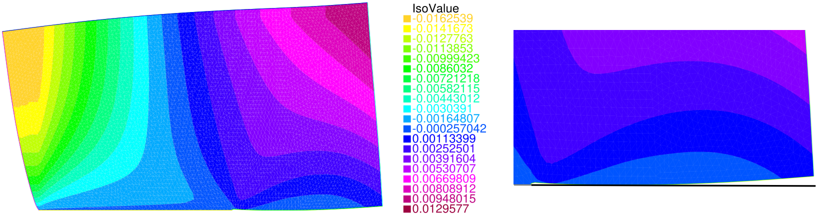
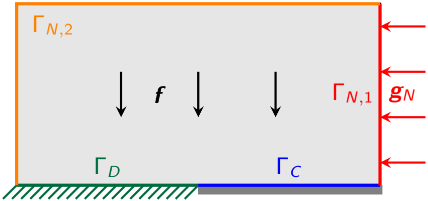
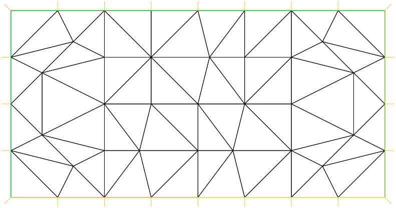
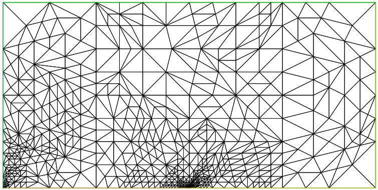


Figure: Vertical displacement in the deformed domain (amplification factor = 5): whole domain (left) and zoom near the contact boundary (right).

Adaptive mesh refinement

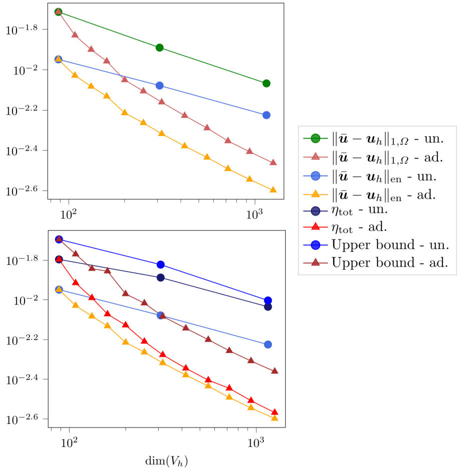


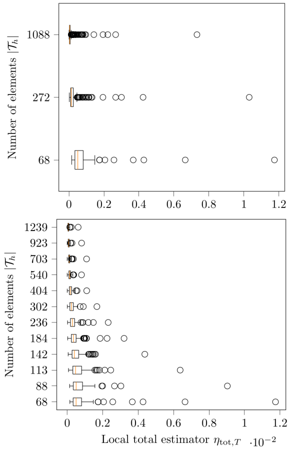




Adaptive VS Uniform refinement

$$\|\boldsymbol{v}\|_{\text{en}} := (\boldsymbol{\sigma}(\boldsymbol{v}), \varepsilon(\boldsymbol{v}))$$

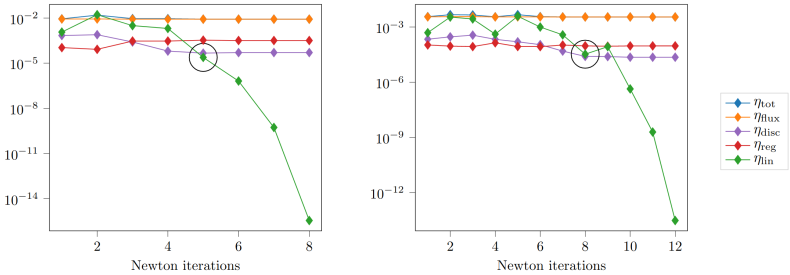




Stopping criteria

	Initial	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th
N_{reg}	7	0	1	0	0	0	0	0	0	0	0	0
N_{lin}	26	2	4	5	3	4	4	4	5	8	8	7

Table: Number of regularization iterations N_{reg} and Newton iterations N_{lin} at each refinement step of the adaptive algorithm with the stopping criteria.



Conclusions:

- Nitsche-based method applied to the unilateral contact problem without friction.
- Regularization and linearization steps.
- A posteriori estimate of the error measured with a dual norm.
- We distinguish the different error components.
- Better asymptotic convergence with adaptive refinement.

Perspectives:

- Extension to the unilateral problem with friction and bilateral problem.
- Extension to contact problem with cohesive forces.
- Industrial application on hydraulic structures.

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